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Finite Group with a Solvable Maximal Subgroup. (**)

1. - Introduction.

Let G be a finite group all of whose proper subgroups are nilpotent. Then by the famous Schmidt-Iwasawa theorem it follows that the group G is solvable. But what can be said about a finite group G if only one maximal subgroup M of G is nilpotent? The following results are known:

(I) (J. G. Thompson [**12**]). If M has odd order, then G is solvable.

(II) (W. E. Deskins [**3**]). If M has class $\leqslant 2$, then G is solvable.

(III) (Z. Janko [**10**]). If a 2-Sylow subgroup of M is of class $\leqslant 2$, then G is solvable.

The above results lead to following question: What can be said about the finite group G which contains a solvable maximal subgroup M which is p-closed and p-nilpotent, p a prime divisor of the order of M? In the present paper we prove:

Theorem 1. *Let G be a finite group with a solvable maximal subgroup M which is p-closed and p-nilpotent, p and odd prime which divides the order of $M/\mathrm{cor}(M)$. If each maximal subgroup L with $\mathrm{cor}(L) = \mathrm{cor}(M)$ is p-closed, then G is solvable.*

Theorem 2. *Let G be a finite group which contains a solvable maximal subgroup M which is 2-closed and 2-nilpotent. If $M/\mathrm{cor}(M)$ has even order and a 2-Sylow subgroup M_2 of M is of class $\leqslant 2$, then G is solvable.*

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The following analogue of the SCHMIDT-IWASAWA theorem is due to ITÔ [9], and is also proved by HUPPERT in [7].

(IV) If a finite group G has all its proper subgroups p-nilpotent, then either G has a normal p-SYLOW subgroup or G is p-nilpotent. In particular, G is p-solvable and $e_p(G) = 1$ if p divides the order of G.

The following two results of J. S. ROSE [11] are somewhat more general than (IV).

(V) If every proper abnormal subgroup of the finite group G is p-nilpotent, and if in addition either (i) the p-SYLOW subgroups of G are Abelian or (ii) p is an odd prime, then G is p-solvable. Furthermore, there exists in G a normal p-subgroup P_0 (possibly trivial) such that G/P_0 is p-nilpotent. If (i) is satisfied, then either G has a normal p-SYLOW subgroup or is itself p-nilpotent. In any case $e_p(G) \leqslant 2$.

(VI) If every proper self-normalizing subgroup of a finite group G is p-nilpotent, then G is p-solvable and $e_p(G) \leqslant 2$.

The results of ITÔ, HUPPERT and ROSE mentioned above lead to the following question: What can be said about the finite group G which contains only one maximal subgroup M which is p-nilpotent, p a prime factor of the order of M? The following two theorems are established in the present paper.

Theorem 3. *Let the finite group G contain a maximal subgroup M which is p-closed and p-nilpotent, p a prime factor of the order of $M/\mathrm{cor}(M)$. Then G is p-solvable and $e_p(G) \leqslant 2$ if G satisfies either of the following conditions:*

(a) *For $p = 2$, a 2-Sylow subgroup M_2 of M is of class $\leqslant 2$.*

(b) *For p odd, each maximal subgroup L of G with $\mathrm{cor}(L) = \mathrm{cor}(M)$ is p-closed.*

Theorem 4. *Let the finite group G contain a maximal subgroup M which is p-closed and p-nilpotent, p a prime divisor of the order of M. Let M be a Hall subgroup of G. Then G is p-solvable and $e_p(G) \leqslant 2$ if G satisfies one of the following conditions:*

(a) *For $p = 2$, a 2-Sylow subgroup M_2 of M is of class $\leqslant 2$.*

(b) *For p odd, each maximal subgroup L with $\mathrm{cor}(L) = \mathrm{cor}(M)$ is p-closed.*

2. - Preliminaries.

The only groups considered here are finite.

If H is a subgroup of the group G, then:

H' is the derived subgroup of H,
$o(H)$ is the order of H,
$[G:H]$ is the index of H in G,
$H^x = x^{-1}Hx$ for each $x \in G$,
$\{H, H^x\}$ is the subgroup of G generated by H and H^x,
$N_G(H)$ is the normalizer of H in G,
$\mathrm{cor}(H) = \bigcap_{x \in G} H^x$ is the core of H in G,
$\varphi(H)$ is the FRATTINI subgroup of H,
$z(H)$ is the center of H.

If x is an element of the group G, then $o(x)$ denotes the order of x.

The subgroup H of the group G is called *self-normalizing* if $N_G(H) = H$. Further, H is called ***abnormal*** if, for each $g \in G$, we have $g \in \{H, H^g\}$. The reader should consult [11] for some of the interesting properties of abnormal subgroups. We note that a maximal subgroup of a group G is either normal or abnormal. Thus the abnormal maximal subgroups of G are precisely its non-normal maximal subgroups (see [11]).

Let p be a prime number. Throughout the present paper $\varphi_p(G)$ denotes the intersection of all maximal subgroups of G whose indices are not divisible by p. W. E. DESKINS [3] showed that $\varphi_p(G)$ is a normal solvable subgroup of G. This fact is useful in proving the results of this paper.

Let π be a nonempty set of prime numbers. Then P_π will denote the set of prime numbers not in π. A positive integer n is called a ***π-number*** if the only prime factors of n are contained in π. An element x in the group G is called a ***π-element*** if $o(x)$ is a π-number. The group G is called a ***π-group*** if $o(G)$ is a π-number. The group G is ***π-closed*** if the products of π-elements of G are π-elements (see [1], [2]). The group G is π-closed if and only if G contains a normal HALL π-subgroup. Subgroups and homomorphic images of π-closed groups are π-closed.

Let p be a prime number. The group G is ***p-closed*** if G possesses a normal p-SYLOW subgroup. The group G is said to be ***p-nilpotent*** if G is P_p-closed. Let P be a p-SYLOW subgroup of G. Then G is called ***p-normal*** if $Z(P)$ is the center of every p-SYLOW subgroup of G in which it is contained.

The following result is used in the proofs of several of our theorems.

Theorem of GRÜN-WIELANDT - P. HALL [5]. Let G be a p-normal group and $Z(P)$ the center of a p-SYLOW subgroup P of G. Then G is p-nilpotent if and only if $N_G(Z(P))$ is p-nilpotent.

The group G is called ***p-solvable*** if each of its composition factors are either a p-group or a P_p-group (see [6]). Thus the group G is p-solvable if and only if G has a series of normal subgroups

$$1 = G_0 < G_1 < \dots < G_n = G \tag{1}$$

for which each factor G_{i+1}/G_i is either a p-group or a P_p-group. The p-length of a p-solvable group G, denoted by $e_p(G)$, is the smallest number of p-factors occurring in any series such as (1) (see [6], [11]).

3. - Basic Lemmas.

In the present section we present three lemmas which will be used in establishing the four theorems mentioned previously. We begin with

Lemma 1. *Let the group G contain a maximal subgroup M which is p-closed and p-nilpotent, p an odd prime factor of $o(M)$. If* cor(M) $= 1$, *then G is p-nilpotent.*

Proof. Let P be a p-Sylow subgroup of M. Since $\text{cor}(M) = 1$ and M is p-closed, it follows that $N_G(P) = M$. Assume by way of contradiction that P is not a p-Sylow subgroup of G, and let Q be a p-Sylow subgroup of G containing P. Then $N_Q(P)$ properly contains P and this contradicts the fact that $N_G(P) = M$. Hence, P is a p-Sylow subgroup of G and $N_G(P) = M$.

For each $x \in G$ let f_x denote the inner automorphism of G induced by x. Let $A = \{f_x \mid x \in N_G(P)\}$ and note that A is a subgroup of the group of automorphisms of G. Let P_1 be a non-trivial A-invariant subgroup of P. Then P_1 is a normal p-subgroup of $N_G(P) = M$. Since $\text{cor}(M) = 1$, it follows that $N_G(P_1) = M$ so that $N_G(P_1)$ is p-nilpotent. Therefore, the elements of $N_G(P_1)$ whose order is prime to p centralize P_1. Because of Theorem A of Thompson [12] it follows that G is p-nilpotent.

Lemma 2. *Let the group G contain a solvable maximal subgroup M of core 1 which is p-closed and p-nilpotent, p an odd prime factor of $o(M)$. If each maximal subgroup of core 1 is p-closed, then G is solvable.*

Proof. Because of Lemma 1 G is p-nilpotent so that we can assume that G is not simple. Let H be a minimal normal subgroup of G. Since $G = HM$, G/H is solvable and we can assume that H is the unique minimal normal subgroup of G.

Let K denote the normal p-complement of G. Then K contains H so that H is a P_p-group. Let L denote a maximal subgroup of G which does not contain H.

Then $[G:L]$ is a P_p-number so that L contains a p-Sylow subgroup P of

G. Since L is p-closed it follows that $N_G(P) = L$. As in the proof of Lemma 1, M contains a p-SYLOW subgroup of G so there exists an element x of G such that $M = N_G(P^x)$. From this it follows that $L^x = M$ so that each maximal subgroup of G which does not contain H is conjugate to M.

Let q be a prime divisor of $[G:M]$ and let R be a maximal subgroup of G such that $[G:R]$ is not divisible by q. Then R is not conjugate to M so that R contains H. Hence, H is contained in $\varphi_q(G)$. By Theorem 2 of [3], H is solvable. Since H and G/H are solvable we conclude that G is solvable.

This completes the proof.

Lemma 3. *Let G be a group with a maximal subgroup of even order which is 2-closed and 2-nilpotent. If* $\mathrm{cor}(M) = 1$ *and 2-Sylow subgroup M_2 of M is of class $\leqslant 2$, then G is 2-nilpotent, hence G is solvable.*

Proof. As in the proof of Lemma 1, M_2 is a 2-SYLOW subgroup of G and $N_G(M_2) = M$. We distinguish two cases.

Case 1. G is 2-normal. Since $\mathrm{cor}(M) = 1$, it follows that $N_G(Z(M_2)) = M$ which is 2-nilpotent. Because of the GRÜN-WIELANDT-P. HALL Thoerem, G is 2-nilpotent.

Case 2. G is not 2-normal. Then there exists an element x of G such that $Z(M_2)$ is nonnormal subgroup of M_2^x and $M_2^x \neq M_2$. Let $D = M_2^x \cap M_2$ and since M_2 is of class $\leqslant 2$ it follows that $M_2' \subseteq Z(M_2)$. Thus D is a normal subgroup of M_2 and since M is 2-nilpotent it follows that D is a normal subgroup of M. Since M is a maximal subgroup of G and $\mathrm{cor}(M) = 1$, $N_G(D) = M$. We note that D is properly contained in M_2^x, hence $N_G(D) \cap M_2^x$ properly contains D. Therefore, there exists an element $y \in N_G(D) \cap M_2^x$ such that $y \notin D$. Since $M_2^x \neq M_2$ and M_2 is normal in M, it follows that $y \notin M$. But $N_G(D) = M$ and so we have a contradiction. Hence, G must be 2-normal.

Let K be a normal 2-complement of G. Then K is of odd order so that by the FEIT- THOMPSON theorem (see [4]) K is solvable. Since G/K is a 2-group, it follows that G is solvable.

4. - Proof Theorem 1.

Because of Lemma 2 we can assume that $\mathrm{cor}(M) \neq 1$. Then $M/\mathrm{cor}(M)$ is a maximal subgroup of $G/\mathrm{cor}(M)$ which satisfies the hypotheses of Lemma 2. Hence by induction on $o(G)$, it follows that $G/\mathrm{cor}(M)$ is solvable. Since $\mathrm{cor}(M)$ is solvable, G is solvable.

5. - Proof of Theorem 2.

By Lemma 3 we can assume that $\mathrm{cor}(M) \neq 1$. Then $M/\mathrm{cor}(M)$ is a maximal subgroup of $G/\mathrm{cor}(M)$ that satisfies all the hypotheses of Lemma 3, hence $G/\mathrm{cor}(M)$ is solvable. Since $\mathrm{cor}(M)$ is solvable, G is solvable.

6. - Proof of Theorem 3.

Because of Lemma 1 and Lemma 3 we can assume that $\mathrm{cor}(M) \neq 1$. We note that $M/\mathrm{cor}(M)$ is a maximal subgroup of $G/\mathrm{cor}(M)$ whose core is 1 and it satisfies the hypotheses of the Theorem. From Lemma 1 and Lemma 3 it follows that $G/\mathrm{cor}(M)$ is p-nilpotent. Being a subgroup of M, $\mathrm{cor}(M)$ is p-closed and p-nilpotent. Hence, it follows that G is p-solvable and $e_p(G) \leqslant 2$.

7. - Proof of Theorem 4.

From Lemma 1 and Lemma 3 we can assume that $\mathrm{cor}(M) \neq 1$. If p divides the order of $M/\mathrm{cor}(M)$, then the Theorem is a consequence of Theorem 3. Assume that p is not a factor of $o(M/\mathrm{cor}(M))$. Then $\mathrm{cor}(M)$ contains a normal p-Sylow subgroup of G. Hence, we conclude that G is p-solvable and $e_p(G) \leqslant 2$.
This completes the proof.

8. - Examples.

Example 1. Let G denote the projective linear group $PSL\ (2,\ 17)$. Then G is a simple group with a 2-Sylow subgroup K of class 3. Moreover, K is a maximal subgroup of G (see [**8**], p. 447). We note that K is a solvable maximal subgroup of G which is 2-closed and 2-nilpotent. Hence, we can not remove the hypothesis in Theorems 2, part (a) in Theorem 3, and part (a) in Theorem 4 of the present paper that a 2-Sylow subgroup M_2 of M is of class $\leqslant 2$.

Example 2. Let $H = GL\ (3, 2)$, the general linear group of 3×3 matrices over the field of two elements. Then H is a simple group of order 168. We recall that H contains a solvable maximal subgroup K which is isomorphic to S_4, the symmetric group on four symbols (see [11], p. 352), is not 2-closed and not 2-nilpotent, but a 2-SYLOW subgroup of K is of class 2. We also note that K is a HALL subgroup of H. Hence, we can not remove the hypothesis in Theorem 2, Theorem 3 and Theorem 4 that M is 2-closed and 2-nilpotent.

Let P be a 7-SYLOW subgroup of H. Then $N_H(P) = L$ is a maximal subgroup of H which is nonnilpotent of order 21. Hence, L is 7-closed but not 7-nilpotent. L is a HALL subgroup of H and each proper subgroup of H is 7-closed (see [11], p. 352). Hence, we can not remove the hypothesis in Theorems 1, 3 and 4 that M is p-nilpotent, p an odd prime factor of $o(M/\mathrm{cor}(M))$.

The mapping $f\colon x \to (x^{-1})^T$ of H onto itself (where, for any $y \in H$, y^{-1} is the inverse of y in H and y^T is the transpose of y in H) is an automorphism of H of order 2. Form the subgroup $G = H\{f\}$ of the holomorph of H, that is, G is a split extension of H by f. Then H is the only normal maximal subgroup of G and each other maximal subgroup of G is supersolvable (see [11], p. 352). Let P be a 7-SYLOW subgroup of G. Then $N = N_G(P)$ is a self-normalizing maximal subgroup and $o(N) = 42$. We note that N is not 2-closed, but N is 2-nilpotent. A 2-SYLOW subgroup of N is Abelian, hence of class $\leqslant 2$. We also note that G is not 2-solvable, but all of the proper abnormal subgroups of G are 2-nilpotent. Therefore, we can not remove the hypothesis in Theorem 2, Theorem 3 and Theorem 4 that M is 2-closed.

Example 3. Let G denote the symmetric group on four symbols and let P be a 2-SYLOW subgroup of G. Then $M = N_G(P) = P$ is a maximal subgroup of G which is 2-closed and 2-nilpotent. We also note that P is of class 2. By Corollary 3 of ([1], p. 138), G satisfies the conditions part (a) of Theorem 3. We also note that $e_2(G) = 2$.

References.

[1] R. BAER, *Classes of finite groups and their properties*, Illinois J. Math. **1** (1957), 115-187.

[2] R. BAER, *Closure and dispersion of finite groups*, Illinois J. Math. **2** (1958), 619-640.

[3] W. E. DESKINS, *A condition for the solvability of a finite group*, Illinois J. Math. **5** (1961), 306-313.

[4] W. FEIT and J. G. THOMPSON, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 755-1029.

[5] M. HALL, *Theory of Groups*, MacMilland Co., New York 1959.

[6] P. HALL and G. HIGMAN, *On the p-length of p-solvable groups and reduction theorems problems for the Burnside's problem*, Proc. London Math. Soc. **6** (1956), 1-42.

[7] B. HUPPERT, *Subnormale Untergruppen und p-Sylow gruppen*, Acta Sci. Math. Szeged **22** (1961), 46-61.

[8] B. HUPPERT, *Endliche Gruppen* (I), Springer-Verlag, New York 1967.

[9] N. ITÔ, *Note on (LM)-groups of finite orders*, Kôdai Math. Sem. Rep. **3** (1951), 1-6.

[10] Z. JANKO, *Finite groups with a nilpotent maximal subgroup*, J. Austral. Math. Soc. **4** (1964), 449-451.

[11] J. S. ROSE, *The influence on a finite group of its proper abnormal structure*, J. London Math. Soc. **40** (1965), 348-361.

[12] J. G. THOMPSON, *Normal p-complements of finite groups*, Math. Z. **72** (1960), 332-354.

Summary.

In the present paper we determine several sufficient conditions for a finite group G to be solvable. Let G contain a solvable maximal subgroup M which is p-closed and p-nilpotent, p an odd prime which divides the order of $M/\mathrm{cor}(M)$. If each maximal subgroup L with $\mathrm{cor}(L) = \mathrm{cor}(M)$ is p-closed, then G is solvable. Also, let G contain a solvable maximal subgroup M which is 2-closed and 2-nilpotent. If $M/\mathrm{cor}(M)$ has even order and a 2-Sylow subgroup M_2 of M is of class $\leqslant 2$, then G is solvable.

Riassunto.

Si determinano condizioni sufficienti perchè un gruppo finito sia risolubile. Il gruppo G contenga un sottogruppo M massimale risolubile che sia p-chiuso e p-nilpotente, essendo p un numero primo dispari che divide l'ordine del quoziente rispetto al suo cuore. Allora se ogni sottogruppo massimale L il cui cuore coincide col cuore di M è p-chiuso, allora G è risolubile.

Il gruppo G contenga un sottogruppo massimale risolubile M che sia 2-chiuso e 2-nilpotente. Se il quoziente di M rispetto al suo cuore ha ordine pari ed un 2-sottogruppo di Sylow M_2 di M ha classe al più due, il gruppo G è risolubile.

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