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EUTIQUIO C. YOUNG (*)

Basic Sets of Polynomials for a Generalized Heat Equation and its Iterates. (**)

1. - Introduction.

Basic sets of polynomial solutions for the wave and LAPLACE's equations and their iterates have been given in a number of papers [1], [2], [3], [4], [5], [6], utilizing various techniques. Some of these results, particularly those of [5] and [6], proved useful in the GRAM-SCHMIDT orthonormalization technique initiated by DAVIS and RABINOWITZ [7] for obtaining approximate solutions of certain boundary value problems. In this paper we propose to develop basic sets of polynomial solutions for the partial differential equations

$$\mathrm{L}^k u \equiv (\mathrm{D}_t - \sum_{i=1}^{m} \mathrm{D}_i)^k u = 0 \qquad (k = 1, 2, \ldots), \tag{1}$$

where

$$\mathrm{D}_t = \frac{\partial}{\partial t}, \qquad \mathrm{D}_i = \frac{\partial^2}{\partial x_i^2} + \frac{\alpha_i}{x_i} \frac{\partial}{\partial x_i},$$

with $\alpha_i \geqslant 0$ $(i = 1, 2, \ldots, m)$. When all the α_i are zero, the differential operator L in (1) reduces to the heat operator

$$H = \mathrm{D}_t - \Delta, \qquad \Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}.$$

Our procedure here is to first establish a basic set of polynomials for the

(*) Indirizzo: Department of Mathematics, Florida State University, Tallahassee, Florida 32306, U.S.A..

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heat equation

$$\mathrm{H}u \equiv u_t - \Delta u = 0 \tag{2}$$

and its iterates, and then adopt the method employed by MILES and YOUNG [8] to develop a basic set for equation (1).

2. - Basic set for $\mathrm{H}^k u = 0$ $(k = 1, 2, \ldots)$.

We state our result as

THEOREM 1. *Let $n \geqslant 1$ and $k \geqslant 1$ be given integers and let $a_1, \ldots, a_m$, s be nonnegative integers satisfying the condition*

$$a_1 + \ldots + a_m = n - 2s\,, \tag{3}$$

with $s \leqslant k-1$ if $n \geqslant 2k$ and $s \leqslant [n/2]$ if $n < 2k$. Then the set P consisting of polynomials homogeneous of degree n in $x_1, \ldots, x_m$, $t^{1/2}$,

$$P^n_{a_1 \ldots a_m s}(x, t) = \sum_{j=0}^{[(n-2s)/2]} \Delta^j(x_1^{a_1} \ldots x_m^{a_m}) \frac{t^{j+s}}{j!}\,, \tag{4}$$

$x = (x_1, \ldots, x_m)$, forms a basic set for the equation

$$\mathrm{H}^k u \equiv (\mathrm{D}_t - \Delta)^k u = 0\,. \tag{5}$$

We shall need the following Lemma which can be easily established by induction.

LEMMA 1. *If $u(x, t)$ satisfies equation (2), then $v(x, t) = t^{k-1} u(x, t)$ satisfies equation (5) for $k = 1, 2, \ldots$.*

We shall prove the Theorem by showing that the set P has correct number of linearly independent polynomial solutions of equation (5). That all the polynomials given by (4) satisfy equation (5) is readily verified. In fact, this follows immediately in the case corresponding to $s = 0$ since the polynomials

$$P^n_{a_1 \ldots a_m 0}(x, t) = \sum_{j=0}^{[n/2]} \Delta^j(x_1^{a_1} \ldots x_m^{a_m}) \frac{t^j}{j!} \tag{6}$$

are easily seen to satisfy equation (2) by direct differentiation with the observation that $\Delta^j(x_1^{a_1} \ldots x_m^{a_m}) = 0$ whenever $j > [n/2]$. In the general case $s > 0$, we observe that the polynomials (4) can be written in the form

$$P^n_{a_1 \dots a_m s}(x, t) = t^s P^{n-2s}_{a_1 \dots a_m 0}(x, t) .$$

Since the polynomials $P^{n-2s}_{a_1 \dots a_m 0}(x, t)$ satisfy the heat equation, it follows by Lemma 1 that all the polynomials (4) satisfy equation (5).

Next we show that the set P is correctly numbered. We observe that P has as many linearly independent polynomials as there are distinct ways of choosing the integers $a_1, \dots, a_m, s$ satisfying the condition (3). Let $n \geqslant 2k$ and suppose that

$$u(x, t) = \sum A_{r_1 \dots r_m \nu}\, x_1^{r_1} \dots x_m^{r_m} t^{\nu} \tag{7}$$

is a polynomial which is homogeneous of degree n in $x_1, \dots, x_m, t^{1/2}$. The summation in (7) is taken over all nonnegative integers r_i and ν such that $r_1 + \dots + r_m + 2\nu = n$. Then every coefficient of u can be represented, except for a constant factor, as

$$A_{r_1 \dots r_m \nu} \sim \mathrm{d}_1^{r_1} \dots \mathrm{d}_m^{r_m} \mathrm{D}_t^{\nu} u , \tag{8}$$

where $\mathrm{d}_i = \partial / \partial x_i$, $1 \leqslant i \leqslant m$. If $\mathrm{H}^k u = 0$, then

$$\mathrm{D}_t^k u \sim \sum \frac{k!}{k_1! \dots k_m! \mu!} \mathrm{d}_1^{2k_1} \dots \mathrm{d}_m^{2k_m} \mathrm{D}_t^{\mu} u,$$

where the summation is taken over all k_i and μ such that $k_1 + \dots + k_m + \mu = k$ with $0 \leqslant \mu \leqslant k-1$. Hence every derivative of the form given in (8) can be written in such a way that D_t occurs no more than $k-1$ times. This means that if the polynomial (7) satisfies equation (5), then all coefficients of u are linear combination of the coefficients $A_{a_1 \dots a_m s}$, where $a_1 + \dots + a_m + 2s = n$ and $0 \leqslant s \leqslant k-1$. Thus the set P is correctly numbered when $n \geqslant 2k$.

In the case when $n < 2k$, a basic set of polynomial solutions for equation (5) could be chosen as simply the set consisting of the monomials $x_1^{a_1} \dots x_m^{a_m} t^s$, where $a_1 + \dots + a_m + 2s = n$. Since $n - 2s \geqslant 0$, it follows that $0 \leqslant s \leqslant [n/2]$ and so in this case, too, P is correctly numbered. Thus the Theorem is proved.

3. - Basic set for $\mathrm{L}^k u = 0$ $(k = 1, 2, \dots)$.

We assume that in (1) at least one of the α_i is not zero and observe that any polynomial solution of equation (1) must be even in x_i whenever $\alpha_i > 0$, $1 \leqslant i \leqslant m$. The latter observation can be proved in similar manner as in [**8**].

Without loss of generality we can assume $\alpha_i = 0$, $1 \leqslant i \leqslant p$, and $\alpha_j > 0$, $p+1 \leqslant j \leqslant m$. Let T_i denote the operator which replaces $x_i^{2s_i}$ by

$$x_i^{(2s_i)} = \frac{1 \cdot 3 \ldots (2s_i - 1)}{(1+\alpha_i) \ldots (2s_i - 1 + \alpha_i)} x_i^{2s_i} ,$$

$p+1 \leqslant i \leqslant m$, and set $T = T_{p+1} \ldots T_m$. We have

Theorem 2. *Let $n \geqslant 1$ and $k \geqslant 1$ be given integers and let $a_1, \ldots, a_p, r_{p+1}, \ldots, r_m, s$ be nonnegative integers such that*

$$\sum_{i=1}^{p} a_i + \sum_{i=p+1}^{m} 2r_i = n - 2s , \tag{9}$$

where $s \leqslant k-1$ if $n > 2k$ and $s \leqslant [n/2]$ if $n < 2k$. Let

$$Q^n_{a_1 \ldots a_p r_{p+1} \ldots r_m s}(x, t) = \sum_{j=0}^{[(n-2s)/2]} \Delta^j (x_1^{a_1} \ldots x_p^{a_p} x_{p+1}^{2r_{p+1}} \ldots x_m^{2r_m}) \frac{t^{j+s}}{j!} .$$

Then the set Q consisting of polynomials

$$R^n_{a_1 \ldots a_p r_{p+1} \ldots r_m s}(x, t) = T Q^n_{a_1 \ldots a_p r_{p+1} \ldots r_m s}(x, t) \tag{10}$$

is a basic set for the equation (1).

Lemma 2. *The differential operators* L *and* H *satisfy the properties* $LT = TH$ *and* $L^k T = TH^k$ *for any integer* $k \geqslant 1$.

We see that $D_t T = T D_t$, $D_i T_i = T_i (\partial^2/\partial x_i^2)$, and $D_i T_j = T_j D_i$ whenever $i \neq j$. Thus

$$LT = \left(D_t - \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^{m} D_i \right) T = T \left(D_t - \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right) - \sum_{i=p+1}^{m} T_{p+1} \ldots D_i T_i \ldots T_m =$$

$$= T \left(D_t - \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right) - \sum_{i=p+1}^{m} T_{p+1} \ldots T_i \frac{\partial^2}{\partial x_i^2} \ldots T_m = T \left(D_t - \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right) - T \sum_{i=p+1}^{m} \frac{\partial^2}{\partial x_i^2} = TH.$$

Repeated application of this result and the use of the associative property $(LT)H = L(TH)$ establishes the second part of the Lemma.

We now prove the Theorem. By Lemma 2 and Theorem 1, we see that all the polynomials given by (10) satisfy equation (1). Hence we need only verify that the set Q contains correct number of linearly independent polynomials which are homogeneous of degree n.

Let $n \geqslant 2k$ and suppose that

(11) $$u(x, t) = \sum A_{s_1 \dots s_p t_{p+1} \dots t_m \nu} x_1^{s_1} \dots x_p^{s_p} x_{p+1}^{(2t_{p+1})} \dots x_m^{(2t_m)} t^{\nu},$$

$s_1 + \dots + s_p + 2t_{p+1} + \dots + 2t_m + 2\nu = n$, $0 \leqslant p \leqslant m-1$, is a polynomial homogeneous of degree n in $x_1, \dots, x_m, t^{1/2}$ which is even in the variables $x_{p+1}, \dots, x_m$. Note that here we have already replaced each $x_i^{2t_i}$ by $x_i^{(2t_i)}$. Then

(12) $$A_{s_1 \dots s_p t_{p+1} \dots t_m \nu} \sim (\mathrm{d}_1^{s_1} \dots \mathrm{d}_p^{s_p} \mathrm{D}_{p+1}^{t_{p+1}} \dots \mathrm{D}_m^{t_m} \mathrm{D}_t^{\nu}) u,$$

so that if $\mathrm{L}^k u = 0$ there follows

$$\mathrm{D}_t^k u \sim \sum B_{k_1 \dots k_m s} (\mathrm{d}_1^{2k_1} \dots \mathrm{d}_p^{2k_p} \mathrm{D}_{p+1}^{t_{p+1}} \dots \mathrm{D}_m^{k_m} \mathrm{D}_t^{s}) u,$$

where the B's are some constants and the summation is taken over all k_i and s such that $k_1 + \dots + k_m + s = k$ with $0 \leqslant s \leqslant k-1$. Thus every derivative of the form (12) can be written in such a way that D_t occurs no more than $k-1$ times. This implies that if $\mathrm{L}^k u = 0$, all coefficients of u are linear combination of the coefficients $A_{a_1 \dots a_p r_{p+1} \dots r_m s}$, where $a_1 + \dots + a_p + 2r_{p+1} + \dots + 2r_m = n - 2s$ and $0 \leqslant s \leqslant k-1$. This shows that the set Q is correctly numbered when $n \geqslant 2k$.

That Q also contains correct number of elements when $n < 2k$ follows from the fact that the set of monomials $x_1^{a_1} \dots x_p^{a_p} x_{p+1}^{2r_{p+1}} \dots x_m^{2r_m} t^s$, where $a_1 + \dots + a_p + 2r_{p+1} + \dots + 2r_m = n - 2s \geqslant 0$, constitutes a basic set of polynomials solutions for equation (1). This estabilishes the Theorem.

4. - Remarks.

It is of interest to note that when $m = 1$ the polynomial solution of degree n of the one-dimensional heat equation $u_t - u_{xx} = 0$ as given in (6) reduces to

(13) $$u_n(x, t) = n! \sum_{j=0}^{[n/2]} \frac{x^{n-2j}}{(n-2j)!} \frac{t^j}{j!}.$$

This coincides with the heat polynomial defined in ([**9**], p. 222) as the coefficient of $z^n/n!$ in the power series expansion

$$\exp(xz + tz^2) = \sum_{n=0}^{\infty} u_n(x, t) \frac{z^n}{n!}.$$

Further, we observe that each element of (6) can be factored as

$$P^n_{a_1 \ldots a_m 0}(x, t) = u_{a_1}(x_1, t) \ldots u_{a_m}(x_m, t) ,$$

where each $u_{a_i}(x_i, t)$ is of the form (13) with n replaced by a_i. Thus according to ([10], pp. 390-391) the polynomials (6) can also be generated as follows

$$\exp [\sum_{i=1}^{m} (x_i z_i + t z_i^2)] = \sum_{a_1=0}^{\infty} \ldots \sum_{a_m=0}^{\infty} P^n_{a_1 \ldots a_m 0}(x, t) \frac{z_1^{a_1}}{a_1!} \ldots \frac{z_m^{a_m}}{a_m!} .$$

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Summary.

Basic sets of polynomial solutions are developed for the heat equation and its iterates. The sets are then extended to from corresponding basic sets for class of generalized heat equations and their iterates.

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