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On a Correspondence Principle in Thermo Linear Viscoelasticity. (**)

1. - Introduction.

A correspondence principle for dynamical problems in the theory of linear viscoelasticity was first enunciated by LEE [4]. However, the thermal response of the material was not included in the derivation of this principle, although viscoelastic materials are affected considerably by temperature. The theory of elasticity considers the effect of temperature and this has led to the study of thermoelasticity. But the solutions to thermal-viscoelastic problems are more difficult; this is due to the fact that the consititutive relations for viscoelastic materials are of very complex type. STERNBERG [5] has formulated a correspondence principle which obtains the solution of a quasi-static problem in thermo linear viscoelasticity from that of a corresponding problem in thermoelasticity. The small deformations were assumed and the physical parameters were also supposed to be independent of temperature. The present derivation of the correspondence principle however extends to the dynamical problems. The body force and the inertia terms are now included in the equation of motion. The integral representation of the stress-strain relation has been used in the present analysis.

An element of length l_i subjected to a temperature difference T becomes $l_i\,(1+\alpha T)$, where α is the coefficient of thermal expansion of the viscoelastic material. The strain ε' due to the free thermal expansion is thus equal to $\alpha T \delta_{ij}$. The total strain ε can be thought to consist of the thermal strain ε' and the viscoelastic strain ε'' produced by the resistence of the medium, i.e.

$$\varepsilon = \varepsilon' + \varepsilon'' . \tag{1}$$

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2. - The constitutive relation and the equations of motion.

The strain ε_{ij} in terms of the displacement components u_i is

$$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{2}$$

Equation of motion is

$$\sigma_{ij,j} + F_i = \varrho\,\frac{\partial^2 u_i}{\partial t^2}\,, \tag{3}$$

where ϱ is the density and F_i is the body force. The strain ε''_{ij} of the viscoelastic deformation is related to the stress σ_{ij} by the stress-strain relation

$$\varepsilon''(t) = E_D^{-1}\left\{\sigma(t) + \int_{-\infty}^{t} \frac{\mathrm{d}\sigma(t')}{\mathrm{d}t'}\,\psi_u(t-t')\,\mathrm{d}t'\right\}$$

or

$$\varepsilon(t) - \alpha T I = E_D^{-1}\left\{\sigma(t) + \int_{-\infty}^{t} \frac{\mathrm{d}\sigma(t')}{\mathrm{d}t'}\,\psi_u(t-t')\,\mathrm{d}t'\right\}. \tag{4}$$

However if we use the following form for stress-strain relation

$$\sigma(t) = E_D\left\{\varepsilon''(t) - \int_{-\infty}^{t} \frac{\mathrm{d}\varepsilon''(t')}{\mathrm{d}t'}\,\varphi_u(t-t')\,\mathrm{d}t'\right\}.$$

We obtain

$$\sigma(t) = E_D\left\{(\varepsilon(t) - \alpha T I) - \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}t'}\left(\varepsilon(t') - \alpha T(t')\, I\right)\varphi_u(t-t')\,\mathrm{d}t'\right\}. \tag{5}$$

The temperature

$$T = T(x_i\,,\ t)\,,$$

E_D = YOUNG's modulus,

ψ_u = Creep function

and

$$\varphi_u = \text{Relaxation function.}$$

Thus the equations in the thermo viscoelasticity are (2), (3) and either (4) or (5). The temperature T is assumed to be known; ordinarily it is determined from the solution of FOURIER's heat equation.

We introduce integral transforms to obtain the algebraic relations connecting the transformed values of stress and strain. Multiplying the equations (4) and (5) by e^{-st}, integrating from $t=0$ to $t=\infty$ and restricting attention to problems for which the solid is undisturbed prior to $t=0$, we have

$$\bar{\varepsilon}(s) - \alpha I \bar{T}(s) = E_D^{-1} [1 + s \bar{\psi}_u(s)] \bar{\sigma}(s) \tag{6}$$

and

$$\bar{\sigma}(s) = E_D [1 - s \bar{\varphi}_u(s)] [\bar{\varepsilon}(s) - \alpha I \bar{T}(s)] , \tag{7}$$

where the convolution theorem is used to evaluate the integral transforms of the integral terms. The equivalence of (6) and (7) implies the following relation between $\bar{\varphi}_u$ and $\bar{\psi}_u$:

$$[1 - s \bar{\varphi}_u(s)] [1 + s \bar{\psi}_u(s)] = 1 \tag{8}$$

which is independent of the thermal response and is same as for the theory of viscoelasticity; an equation due to GROSS [2]. The above relation corelates the creep and the relaxation behaviour of a viscoelastic solid. The relation connecting the ultimate values of creep and relaxation functions may be obtained by using an Abelian theorem and with the limit $s \to 0$, giving [3]

$$\psi_u(\infty) = \frac{\varphi_u(\infty)}{1 - \varphi_u(\infty)} . \tag{9}$$

Either of the forms (6) or (7) may be written as

$$\bar{\sigma}(s) = E(s) [\bar{\varepsilon}(s) - \alpha I \bar{T}(s)] \tag{10}$$

where

$$E(s) = E_D [1 - s \bar{\varphi}_u(s)] = E_D [1 + s \bar{\psi}_u(s)]^{-1} . \tag{11}$$

E_D is a function of complex variable s. It may be regarded as giving a complete specification of the viscoelastic properties of the solid. Also the LAPLACE transform of the equation of motion (3) gives

(12) $$\bar{\sigma}_{ij,j} + \bar{F}_i = \varrho\, s^2\, \bar{u}_i(s)$$

the continuum is assumed initially at rest. If ψ_1 and ψ_2 are the creep functions governing the shear and dilatational behaviour and μ, k denote the shear modulus and bulk modulus respectively. Then the stress-strain relations (4) can also be written as

(13) $$2\mu\, e_{ij} = s_{ij} + \int_{-\infty}^{t} \frac{\mathrm{d}s_{ij}(t')}{\mathrm{d}t'}\, \psi_1(t-t')\, \mathrm{d}t'$$

and

(14) $$3\, k\, (\varepsilon_{ii} - 3\alpha T) = \sigma_{ii} + \int_{-\infty}^{t} \frac{\mathrm{d}\sigma_{ii}(t')}{\mathrm{d}t'}\, \psi_2(t-t')\, \mathrm{d}t' ,$$

where $\sigma_{ii}/3$ is the mean normal stress and $\varepsilon_{ii}/3$ is the mean extension and σ_{ij}, e_{ij} are stress and strain deviators respectively; so that

(15) $$\begin{cases} s_{ij} = \sigma_{ij} - \frac{1}{3}\, \sigma_{kk}\, \delta_{ij} \\ e_{ij} = \varepsilon_{ij} - \frac{1}{3}\, \varepsilon_{kk}\, \delta_{ij} . \end{cases}$$

Applying LAPLACE transformation to (13) and (14), we have

(16) $$2\, \mu\, \bar{e}_{ij} = \bar{s}_{ij}(s) + s\, \bar{\psi}_1(s)\, \bar{s}_{ij}(s)$$

and

(17) $$3\, k\, (\bar{\varepsilon}_{ii} - 3\alpha\bar{T}) = \bar{\sigma}_{ii}(s) + s\, \bar{\psi}_2(s)\, \bar{\sigma}_{ii}(s)$$

which give

(18) $$\begin{cases} \bar{s}_{ij}(s) = \dfrac{1}{1 + s\, \bar{\psi}_1(s)}\, 2\mu\, \bar{e}_{ij} \\ \bar{\sigma}_{ii}(s) = \dfrac{1}{1 + s\, \bar{\psi}_2(s)}\, [3\, k\, (\bar{\varepsilon}_{ii} - 3\alpha\bar{T})] . \end{cases}$$

From (15) and (18), we get

$$\bar{\sigma}_{ij} = 2\,\mu_1\,\bar{\varepsilon}_{ij} + \left(k_1 - \frac{2}{3}\,\mu_1\right)\bar{\varepsilon}_{kk}\,\delta_{ij} - 3\,k_1\,\alpha\,\bar{T}\,\delta_{ij}\,, \tag{19}$$

where

$$\mu_1(s) = \mu/(1 + s\,\bar{\psi}_1(s))$$

and

$$k_1(s) = k/(1 + s\,\bar{\psi}_2(s))\,. \tag{20}$$

It is easily seen that the equation in thermoelasticity corresponding to (19) is of the form

$$\bar{\sigma}_{ij} = 2\,\mu\,\bar{\varepsilon}_{ij} + \left(k - \frac{2}{3}\,\mu\right)\bar{\varepsilon}_{kk}\,\delta_{ij} - 3\,k\,\alpha\,\bar{T}\,\delta_{ij}\,. \tag{21}$$

Hence it follows that in the LAPLACE transform plane the components of the stress for thermoviscoelastic problems are obtainable from those of the thermoelastic problems of same type by replacing k and μ by $k_1(s)$ and $\mu_1(s)$ respectively.

Substituting $\bar{\sigma}_{ij}$ from (19) into (12), we obtain

$$\mu_1\,\nabla^2\,\bar{u} + \left(k_1 + \frac{1}{3}\,\mu_1\right)\bar{u}_{j,ji} + \bar{F}_i - 3\,k_1\,\alpha\,\bar{T}_i = \varrho\,s^2\,\bar{u}_i(s). \tag{22}$$

(22) can now be compared with the thermal equations in elasticity

$$\mu\,\nabla^2\,\bar{u} + \left(k + \frac{1}{3}\,\mu\right)\bar{u}_{j,ji} + \bar{F}_i - 3\,k\,\alpha\,\bar{T}_i = \varrho\,s^2\,\bar{u}_i(s)\,. \tag{23}$$

The following conclusion can now be drawn:

« The LAPLACE transform of the solution of a thermo linear viscoelastic problem can be obtained from the solution of the LAPLACE transform of the equations giving the displacements in a thermoelastic problem with same boundary and initial conditions, by replacing k and μ by k_1 and μ_1 respectively. » The solution of the problem as such is then obtained by inverting the transforms. The equations (23) are to be solved subject to the specified displacements u_i or tractions σ_i on the surface of the body. The effect of temperature change T is equivalent to replacing the body force $\bar{F}_i$ by $\bar{F}_i - 3\,k_1\,\alpha\,\bar{T}_{,i}$ and to substituting $\bar{\sigma}_i + 3\,k_1\,\alpha\,\bar{T}\,\nu_i$ for the surface tractions in boundary conditions. The additional term $3k_1\,\bar{T}\,\alpha\,\nu_i$ being equivalent to a hydrostatic pressure.

References.

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Abstract.

A correspondence principle for dynamical problems in the theory of thermo linear viscoelasticity has been established, which enables one to write the solution of a thermo-viscoelastic problem from the solution of an identical problem in thermo-elasticity. The consequences of the temperature variation on the thermal stresses and the body force are also observed.

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