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Expansions of the Hypergeometric Functions of Three Variables. (**)

Certain hypergeometric functions of three variables have been defined by SARAN [2]. In a paper that has appeared in this journal [3], SARAN has obtained some integral representation of LAPLACE type for these hypergeometric functions. The object of this paper is to derive some expansions involving these hypergeometric functions of three variables, with the help of the results given by SARAN [3]. These expansions are the generalisations of the results for the APPELL's hypergeometric function recently obtained by the author [1].

These expansions have been obtained in section B, with the help of the three lemmas, obtained by the author [1], given in section A, about the transform defined by the integral equation

$$(1.1)\qquad \psi_{\lambda,k,m}(p) = \int_0^\infty e^{-(1/2)pt}\,(pt)^{\lambda-(1/2)}\,M_{k,m}(pt)\,f(t)\,dt\,.$$

Later on in section C, another recurrence relations for the APPELL's hypergeometric functions, which are the particular cases of the results obtained in section B, have also been obtained.

Here we shall denote $f(t)\in c(\sigma,\ \varrho;\ \alpha)$, if

$$f(t) = \begin{cases} O(t^{\sigma}) & \text{for small } t \\ O(e^{\alpha t}\, t^{\varrho}) & \text{for large } t. \end{cases}$$

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Section A.

Lemma I. *If*

$$f(t) \in c(\sigma, \varrho; \alpha), \qquad \mathrm{Re}(\lambda + \sigma + m + 1) > 0, \qquad \mathrm{Re}\, p > \mathrm{Re}\, \alpha,$$

then

$$(2.1) \quad \left\{ \begin{aligned} &\sum_{r=0}^{n} (-1)^r \, n_{c_r} \frac{\Gamma(\frac{1}{2} - k - m)\, \Gamma(m - k + n - r + \frac{1}{2})\, \Gamma(2m+1)}{\Gamma(\frac{1}{2} - k - m - r)\, \Gamma(2m + n + 1)\, \Gamma(m - k + \frac{1}{2})} \cdot \\ &\qquad \cdot \psi_{\lambda - \frac{1}{2}n,\, k - \frac{1}{2}n + r,\, m + \frac{1}{2}n}(p) = \psi_{\lambda, k, m}(p) . \end{aligned} \right.$$

Lemma II. *If*

$$f(t) \in c(\sigma, \varrho; \alpha), \qquad \mathrm{Re}(\lambda + \sigma + m + 1) > 0, \qquad \mathrm{Re}\, p > \mathrm{Re}\, \alpha,$$

then

$$(2.2) \quad \left\{ \begin{aligned} &\sum_{r=0}^{n} (-1)^{n-r} \, n_{c_r} \frac{\Gamma(k + m + n - r + \frac{1}{2})\, \Gamma(m - k - n + \frac{1}{2})\, \Gamma(2m + 1)}{\Gamma(k + m + \frac{1}{2})\, \Gamma(2m + 1 - r)\, \Gamma(m - k + \frac{1}{2})} \cdot \\ &\qquad \cdot \psi_{\lambda + \frac{1}{2}r,\, k + n - \frac{1}{2}r,\, m - \frac{1}{2}r}(p) = \psi_{\lambda, k, m}(p) . \end{aligned} \right.$$

Lemma III. *If*

$$f(t) \in c(\sigma, \varrho; \alpha), \qquad \mathrm{Re}(\lambda + \sigma + m + 1) > 0, \qquad \mathrm{Re}\, p > \mathrm{Re}\, \alpha,$$

then

$$(2.3) \quad \left\{ \begin{aligned} &\sum_{r=0}^{n} (-1)^{n-r} \, n_{c_r} \frac{\Gamma(m - k + n - r + \frac{1}{2})\, \Gamma(m + k - n + \frac{1}{2})\, \Gamma(2m + 1)}{\Gamma(2m + 1 - r)\, \Gamma(m + k + \frac{1}{2})\, \Gamma(m - k + \frac{1}{2})} \cdot \\ &\qquad \cdot \psi_{\lambda + \frac{1}{2}r,\, k - n + \frac{1}{2}r,\, m - \frac{1}{2}r}(p) = \psi_{\lambda, k, m}(p) . \end{aligned} \right.$$

Section B.

In this section, we obtain some expansions for the hypergeometric function of three variables, with the help of the lemmas given in section A.

If we take $f(t) = e^{-at}\, \psi_2(b_2;\; c_2, c_3;\; yt, zt)$, then it can be easily derived from the SARAN's result ([3], p. 134) that

$$\psi_{\lambda, k, m}(p) = \frac{\Gamma(\lambda+m+1)}{a^{m+\lambda+1}}\, p^{m+\lambda} \cdot$$

$$\cdot\, F_E\Big[\lambda+m+1,\; \lambda+m+1,\; \lambda+m+1,\; \tfrac{1}{2}+k+m,\; b_2,\; b_2;\; 1+2m,\; c_2,\; c_3;\; -\frac{p}{a},\; \frac{y}{a},\; \frac{z}{a}\Big]$$

valid for $\mathrm{Re}(\lambda+m+1)>0$ and $\mathrm{Re}(p+a) > \mathrm{Re}(\sqrt{y}+\sqrt{z})^2$.

On using this value of $\psi_{\lambda, k, m}(p)$ in the results (2.1), (2.2) and (2.3) respectively, we get the following relations after making suitable adjustment in parameters.

$$(3.1)\quad \begin{cases} \displaystyle\sum_{r=0}^{n} (-1)^r\, n_{c_r} \frac{\Gamma(1-b_1)\,\Gamma(c_1-b_1+n-r)\,\Gamma(c_1)}{\Gamma(1-b_1-r)\,\Gamma(c_1+n)\,\Gamma(c_1-b_1)}\, \cdot \\ \quad \cdot\, F_E[a_1, a_1, a_1, b_1+r, b_2, b_2;\; c_1+n, c_2, c_3;\; x, y, z] = \\ \quad = F_E[a_1, a_1, a_1, b_1, b_2, b_2;\; c_1, c_2, c_3;\; x, y, z], \end{cases}$$

$$(3.2)\quad \begin{cases} \displaystyle\sum_{r=0}^{n} (-1)^{n-r}\, n_{c_r} \frac{\Gamma(b_1+n-r)\,\Gamma(c_1-b-n)\,\Gamma(c_1)}{\Gamma(b_1)\,\Gamma(c_1-r)\,\Gamma(c_1-b_1)}\, \cdot \\ \quad \cdot\, F_E[a_1, a_1, a_1, b_1+n-r, b_2, b_2;\; c_1-r, c_2, c_3;\; x, y, z] = \\ \quad = F_E[a_1, a_1, a_1, b_1, b_2, b_2;\; c_1, c_2, c_3;\; x, y, z] \end{cases}$$

and

$$(3.3)\quad \begin{cases} \displaystyle\sum_{r=0}^{n} (-1)^{n-r}\, n_{c_r} \frac{\Gamma(c_1-b_1+n-r)\,\Gamma(b_1-n)\,\Gamma(c_1)}{\Gamma(c_1-r)\,\Gamma(b_1)\,\Gamma(c_1-b_1)}\, \cdot \\ \quad \cdot\, F_E[a_1, a_1, a_1, b_1-n, b_2, b_2;\; c_1-r, c_2, c_3;\; x, y, z] = \\ \quad = F_E[a_1, a_1, a_1, b_1, b_2, b_2;\; c_1, c_2, c_3;\; x, y, z]. \end{cases}$$

Similarly if we take $f(t)$ as $e^{-at}\,\Xi_2(b_2, b_3; c_2; yt, zt)$, $e^{-at}\,\psi_1(b_1, a_1, c_1, c_3; x, zt)$, $e^{-at}\,\Phi_1(a_2, b_2; c_2, zt, y)$ and $e^{-at}\,\Xi_1(a_2, a_3, b_2, c_2; y, zt)$ and write A, B and C for the quantity

$$\frac{\Gamma(1-b_1)\,\Gamma(c_1-b_1+n-r)\,\Gamma(c_1)}{\Gamma(1-b_1-r)\,\Gamma(c_1+n)\,\Gamma(c_1-b_1)}, \qquad \frac{\Gamma(b_1+n-r)\,\Gamma(c_1-b_1-n)\,\Gamma(c_1)}{\Gamma(b_1)\,\Gamma(c_1-r)\,\Gamma(c_1-b_1)},$$

$$\frac{\Gamma(c_1-b_1+n-r)\,\Gamma(b_1-n)\,\Gamma(c_1)}{\Gamma(c_1-r)\,\Gamma(b_1)\,\Gamma(c_1-b_1)},$$

respectively, then, by virtue of the results given by SARAN ([3], p. 134), we get the following relations after making suitable adjustment in parameters:

$$(3.4)\quad \begin{cases} \sum_{r=0}^{n}(-1)^r\, n_{c_r}\, A\cdot F_G[a_1, a_1, a_1, b_1+r, b_2, b_3; c_1+n, c_2, c_2; x, y, z] = \\ \qquad = F_G[a_1, a_1, a_1, b_1, b_2, b_3; c_1, c_2, c_2; x, y, z], \end{cases}$$

$$(3.5)\quad \begin{cases} \sum_{r=0}^{n}(-1)^{n-r}\, n_{c_r}\, B\cdot F_G[a_1, a_1, a_1, b_1+n-r, b_2, b_3; c_1-r, c_2, c_2; x, y, z] = \\ \qquad = F_G[a_1, a_1, a_1, b_1, b_2, b_3; c_1, c_2, c_2; x, y, z], \end{cases}$$

$$(3.6)\quad \begin{cases} \sum_{r=0}^{n}(-1)^{n-r}\, n_{c_r}\, C\cdot F_G[a_1, a_1, a_1, b_1-n, b_2, b_3; c_1-r, c_2, c_2; x, y, z] = \\ \qquad = F_G[a_1, a_1, a_1, b_1, b_2, b_3; c_1, c_2, c_2; x, y, z], \end{cases}$$

$$(3.7)\quad \begin{cases} \sum_{r=0}^{n}(-1)^r\, n_{c_r}\, A\cdot F_K[a_2, a_1, a_1, b_2, b_1+r, b_2, c_2, c_1+n, c_3; x, y, z] = \\ \qquad = F_K[a_2, a_1, a_1, b_2, b_1, b_2; c_2, c_1, c_3; x, y, z], \end{cases}$$

$$(3.8)\quad \begin{cases} \sum_{r=0}^{n}(-1)^{n-r}\, n_{c_r}\, B\cdot F_K[a_2, a_1, a_1, b_2, b_1+n-r, b_2; c_2, c_1-r, c_3; x, y, z] = \\ \qquad = F_K[a_2, a_1, a_1, b_2, b_1, b_2, c_2, c_1, c_3; x, y, z], \end{cases}$$

$$(3.9)\quad \begin{cases} \sum_{r=0}^{n}(-1)^{n-r}\, n_{c_r}\, C\cdot F_K[a_2, a_1, a_1, b_2, b_1-n, b_2, c_2, c_1-r, c_3; x, y, z] = \\ \qquad = F_K[a_2, a_1, a_1, b_2, b_1, b_2; c_2, c_1, c_3; x, y, z], \end{cases}$$

$$(3.10)\begin{cases} \sum_{r=0}^{n} (-1)^r n_{c_r} A \cdot F_M[b_1 + r, a_2, a_2, a_1, b_2, a_1; c_1 + n, c_2, c_2; x, y, z] = \\ \qquad = F_M[b_1, a_2, a_2, a_1, b_2, a_1; c_1, c_2, c_2; x, y, z], \end{cases}$$

$$(3.11)\begin{cases} \sum_{r=0}^{n} (-1)^{n-r} n_{c_r} B \cdot F_M[b_1 + n - r, a_2, a_2, a_1, b_2, a_1; c_1 - r, c_2, c_2; x, y, z] = \\ \qquad = F_M[b_1, a_2, a_2, a_1, b_2, a_1; c_1, c_2, c_2; x, y, z], \end{cases}$$

$$(3.12)\begin{cases} \sum_{r=0}^{n} (-1)^{n-r} n_{c_r} C \cdot F_M[b_1 - n, a_2, a_2, a_1, b_2, a_1; c_1 - r, c_2, c_2; x, y, z] = \\ \qquad = F_M[b_1, a_2, a_2, a_1, b_2, a_1; c_1, c_2, c_2; x, y, z], \end{cases}$$

$$(3.13)\begin{cases} \sum_{r=0}^{n} (-1)^r n_{c_r} A \cdot F_N[b_1 + r, a_2, a_3, a_1, b_2, a_1; c_1 + n, c_2, c_2; x, y, z] = \\ \qquad = F_N[b_1, a_2, a_3, a_1, b_2, a_1; c_1, c_2, c_2; x, y, z], \end{cases}$$

$$(3.14)\begin{cases} \sum_{r=0}^{n} (-1)^{n-r} n_{c_r} B \cdot F_N[b_1 + n - r, a_2, a_3, a_1, b_2, a_1; c_1 - r, c_2, c_2; x, y, z] = \\ \qquad = F_N[b_1, a_2, a_3, a_1, b_2, a_1; c_1, c_2, c_2; x, y, z], \end{cases}$$

$$(3.15)\begin{cases} \sum_{r=0}^{n} (-1)^{n-r} n_{c_r} C \cdot F_N[b_1 - n, a_2, a_3, a_1, b_2, a_1; c_1 - r, c_2, c_2; x, y, z] = \\ \qquad = F_N[b_1, a_2, a_3, a_1, b_2, a_1; c_1, c_2, c_2; x, y, z]. \end{cases}$$

Section C.

In this section, few interesting cases giving relations for the APPELL's hypergeometric function have been given.

(i) If $y \to 0$, then $F_N, F_E \to F_2$ and the results (3.1), (3.2), (3.3), (3.13), (3.14) and (3.15) give us the following results obtained by the author [**1**].

$$(4.1)\begin{cases} \sum_{r=0}^{n} (-1)^r n_{c_r} \frac{\Gamma(1 - b_1)\,\Gamma(c_1 - b_1 + n - r)\,\Gamma(c_1)}{\Gamma(1 - b_1 - r)\,\Gamma(c_1 + n)\,\Gamma(c_1 - b_1)} \cdot \\ \cdot F_2[a_1; b_2, b_1 + r, c_3, c_1 + n; x, y] = F_2[a_1; b_2, b_1, c_3, c_1; x, y], \end{cases}$$

$$(4.2)\quad \left\{\begin{aligned} &\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}c_r \frac{\Gamma(b_1+n-r)\,\Gamma(c_1-b_1-n)\,\Gamma(c_1)}{\Gamma(b_1)\,\Gamma(c_1-r)\,\Gamma(c_1-b_1)}\cdot \\ &\cdot F_2[a_1;\ b_2,\ b_1+n-r,\ c_3,\ c_1-r;\ x,\ y] = F_2[a_1;\ b_2,\ b_1,\ c_3,\ c_1;\ x,\ y]\,, \end{aligned}\right.$$

$$(4.3)\quad \left\{\begin{aligned} &\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}c_r \frac{\Gamma(c_1-b_1+n-r)\,\Gamma(b_1-n)\,\Gamma(c_1)}{\Gamma(c_1-r)\,\Gamma(b_1)\,\Gamma(c_1-b_1)}\cdot \\ &\cdot F_2[a_1;\ b_2,\ b_1-n,\ c_3,\ c_1-r;\ x,\ y] = F_2[a_1;\ b_2,\ b_1,\ c_3,\ c_1;\ x,\ y]\,. \end{aligned}\right.$$

(ii) On taking $n = 1$, results (3.1), (3.2) and (3.3) give the following recurrence relation for

$$(4.4)\quad \left\{\begin{aligned} &(c_1-b_1)\, F_E[a_1,\ a_1,\ a_1,\ b_1,\ b_2,\ b_2,\ c_1+1,\ c_2,\ c_3;\ x,\ y,\ z] + \\ &+ b_1\, F_E[a_1,\ a_1,\ a_1,\ b_1+1,\ b_2,\ b_2,\ c_1+1,\ c_2,\ c_3;\ x,\ y,\ z] = \\ &= c_1\, F_E[a_1,\ a_1,\ a_1,\ b_1,\ b_2,\ b_2,\ c_1,\ c_2,\ c_3;\ x,\ y,\ z]\,. \end{aligned}\right.$$

Similar recurrence relations can also be obtained for F_G, F_K, F_M and F_N, when we take $n = 1$, in other results of Section B.

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References.

[1] M. A. PATHAN, *Certain recurrence relations for the double generalised hypergeometric function* (paper communicated for publication).

[2] S. SARAN, *Hypergeometric functions of three variables*, Ganita **5** (1954), 77-91.

[3] S. SARAN, *Integral representations of Laplace type for certain hypergeometric function of three variables*, Riv. Mat. Univ. Parma (1) **8** (1957), 133-143.

Abstract.

The object of this paper is to obtain some expansions involving hypergeometric function of three variables defined by Saran, which are the generalisation of the expansions for the Appell's hypergeometric function of two variables.

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