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On Meijer Transform. (**)

1. – A generalization of the classical LAPLACE transform

$$\varphi(s) = s \int_0^\infty e^{-st} f(t)\, dt\,, \tag{1}$$

has been given by MEIJER ([1], p. 730) in the form

$$\varphi(s) = s \int_0^\infty e^{-\frac{1}{2}st}\, (st)^{-k-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(st)\, f(t)\, dt\,, \tag{2}$$

where $W_{k,m}(z)$ is the WHITTAKER's confluent hypergeometric function. In this equation $f(t)$ is known as the *original* of $\varphi(s)$, $\varphi(s)$ the *image* of $f(t)$ and (1) is symbolically denoted by (JAISWAL [2], p. 385)

$$f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s)\,,$$

and (1) by

$$f(t) \doteqdot \varphi(s)\,.$$

For $k = -m$, and by virtue of the identity ([3], p. 432)

$$e^{-\frac{1}{2}st} \equiv (st)^{m-\frac{1}{2}}\, W_{-m+\frac{1}{2},m}(st)\,,$$

(2) reduces to (1).

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In this paper we have studied two properties of MEIJER transform. One is the generalization of a known result of LAPLACE transform given by BOSE ([4], p. 821). The other property involves the images of MEIJER transforms, which are self-reciprocal in the HANKEL transform. These properties have been illustrated by suitable examples.

2. - Theorem I.

If $f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s)$, *then*

$$\int_1^\infty \varphi(s)\, h(s)\, ds = \int_0^\infty t^{-k-\frac{1}{2}} f(t)\, \psi(t)\, dt\,, \tag{3}$$

where

$$\psi(t) = \int_1^\infty e^{-\frac{1}{2}st} s^{-k+\frac{1}{2}} W_{k+\frac{1}{2},m}(st)\, h(s)\, ds\,, \tag{4}$$

provided the integrals involved are absolutely convergent.

Proof. We have

$$\varphi(s) = s \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(st)\, f(t)\, dt\,.$$

Multiplying both the sides by $h(s)$ and integrating with respect to s between the limits *one* and *infinity*, we have

$$\int_1^\infty \varphi(s)\, h(s)\, ds = \int_1^\infty s\, h(s) \left[\int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(st)\, f(t)\, dt\right] ds$$

$$= \int_0^\infty t^{-k-\frac{1}{2}} f(t) \left[\int_1^\infty s^{-k+\frac{1}{2}} e^{-\frac{1}{2}st} W_{k+\frac{1}{2},m}(st)\, h(s)\, ds\right] dt$$

$$= \int_0^\infty t^{-k-\frac{1}{2}} f(t)\, \psi(t)\, dt\,,$$

where $\psi(t)$ is given by (4).

The change of order of integration follows from the absolute convergence of the integrals.

2.1. - Corollary.

If we take $k = -m$ in the above Theorem, we get a known result in Operational Calculus ([4], p. 821):

If $f(t) \doteqdot \varphi(s)$, then

$$\int^{\infty} \varphi(s)\, h(s)\, \mathrm{d}s = \int_0^{\infty} f(t)\, \psi(t)\, \mathrm{d}t\,,$$

where

$$\psi(t) = \int_1^{\infty} s\, e^{-st}\, h(s)\, \mathrm{d}s\,,$$

provided the integrals converge absolutely.

2.2. - Example.

Let

$$\varphi(s) = s^{-\varrho}\,.$$

Then (Jaiswal (1) [2], p. 387),

$$f(t) = \frac{\Gamma(\varrho - 2k + 1)}{\Gamma_*(\varrho - k + 1 \pm m)}\, t^{\varrho},$$

$$\mathrm{Re}(\varrho - k) + 1 > |\,\mathrm{Re}\, m\,| \qquad \text{and} \qquad \mathrm{Re}\, s > 0.$$

Further, let

$$h(s) = (s-1)^{\sigma-1}\, G^{l,n}_{p,q}\left(as \,\middle|\, \begin{matrix} a_1\,, \ldots, a_p \\ b_1\,, \ldots, b_q \end{matrix}\right).$$

Then, from (4), we have

$$\psi(t) = \int_1^{\infty} e^{-\frac{1}{2}st}\, s^{-k+\frac{1}{2}}\, W_{k+\frac{1}{2},m}(st)\, (s-1)^{\sigma-1}\, G^{l,n}_{p,q}\left(as \,\middle|\, \begin{matrix} a_1\,, \ldots, a_p \\ b_1\,, \ldots, b_q \end{matrix}\right) \mathrm{d}s\,.$$

(1) The symbol $\Gamma_*(a \pm b)$ denotes $\Gamma(a+b)\,\Gamma(a-b)$ and the symbol

$${}_mF_n\left\{\begin{matrix} \alpha_1 \pm \beta_1,\ \alpha_2 \pm \beta_2,\ \ldots \\ \qquad ;\, z \\ a_1 \pm b_1,\ a_2 \pm b_2,\ \ldots \end{matrix}\right\} \text{ denotes } {}_mF_n\left\{\begin{matrix} \alpha_1+\beta_1,\ \alpha_1-\beta_1,\ \alpha_2+\beta_2,\ \alpha_2-\beta_2,\ldots \\ \qquad ;\, z \\ a_1+b_1,\ a_1-b_1,\ a_2+b_2,\ a_2-b_2,\ldots \end{matrix}\right\}.$$

Hence, from (3), we get

$$\frac{\Gamma(\varrho-2k+1)}{\Gamma_*(\varrho-k+1\pm m)}\int_0^\infty t^{\varrho-k-\frac{1}{2}}\,\psi(t)\,\mathrm{d}t = \tag{5}$$

$$=\int_1^\infty s^{-\varrho}(s-1)^{\sigma-1}\,G^{l,n}_{p,q}\left(as\left|\begin{matrix}a_1,\,\ldots,\,a_p\\ b_1,\,\ldots,\,b_q\end{matrix}\right.\right)\mathrm{d}s=\Gamma(\sigma)\,G^{l+1,n}_{p+1,q+1}\left(a\left|\begin{matrix}a_1,\,\ldots,\,a_p,\,\varrho\\ b_1,\,\ldots,\,b_q\end{matrix}\right.\right),$$

on using a known integral ([3], pp. 417-418), provided $\mathrm{Re}\, s>0$, $\mathrm{Re}(\varrho-k)+1>|\mathrm{Re}\, m|$ and either of the following set of conditions:

$$\text{(i)}\quad\begin{cases} p+q<2(l+n)\,, \qquad |\arg a|<(l+n-\frac{1}{2}p-\frac{1}{2}q)\,\pi \\ \mathrm{Re}\,(\varrho-\sigma-a_j)+1>0 \qquad (j=1,\,2,\,\ldots,\,n)\,, \qquad \mathrm{Re}\,\sigma>0\,, \end{cases}$$

$$\text{(ii)}\quad\begin{cases} p+q\leqslant 2(l+n)\,, \qquad |\arg a|\leqslant(l+n-\frac{1}{2}p-\frac{1}{2}q)\,\pi \\ \mathrm{Re}\,(\varrho-\sigma-a_j)+1>0 \qquad (j=1,\,2,\,\ldots,\,n)\,, \qquad \mathrm{Re}\,\sigma>0 \\ \mathrm{Re}\,[\sum_{j=1}^{p}a_j-\sum_{j=1}^{q}b_j+(q-p)(\varrho-\sigma+\frac{1}{2})]+\frac{1}{2}>0\,, \end{cases}$$

or

$$\text{(iii)}\quad\begin{cases} q<p \qquad (\text{or } q\leqslant p \text{ and } |a|>1) \\ \mathrm{Re}(\varrho-\sigma-a_j)+1>0 \qquad (j=1,\,2,\,\ldots,\,n)\,, \qquad \mathrm{Re}\,\sigma>0\,. \end{cases}$$

3. - Theorem. II.

If $f(t)\xrightarrow[m]{k+\frac{1}{2}}\varphi(s)$ *and* $\varphi(s)$ *is self-reciprocal in Hankel transform of order* ν, *then*

$$\varphi(s)=\int_0^\infty f(x)\,G_\nu(s/x)\,\mathrm{d}x\,, \tag{6}$$

where

$$\begin{cases} G_\nu(s/x)=\dfrac{\Gamma_*(\nu-k+(5/2)\pm m)}{2^\nu\Gamma\big(\nu-2k+(5/2)\big)\,\Gamma(\nu+1)}\,x^{-2}\,(s/x)^{\nu+\frac{1}{2}}\,\cdot \\ \qquad\cdot\,{}_4F_3\left\{\begin{matrix}\big(\nu-k+(5/2)\pm m\big)/2\,,\;\big(\nu-k+(7/2)\pm m\big)/2 \\ \nu+1,\;\big(\nu-2k+(5/2)\big)/2\,,\;\big(\nu-2k+(7/2)\big)/2\end{matrix}\,;\,-s^2/x^2\right\}, \end{cases} \tag{7}$$

provided that $\mathrm{Re}(\nu + k) + (5/2) > |\mathrm{Re}\, m|$, $\mathrm{Re}(\mu_1 - k) + 1 > |\mathrm{Re}\, m|$, *where* $f(x) = O(x^{\mu_1})$ *for small* x, $f(x)$ *is continuous for* $x \geqslant \varepsilon > 0$, $e^{-sx} f(x) \to 0$ *as* $x \to \infty$ *for* $\mathrm{Re}\, s \geqslant s_0 > 0$ *and the integral in* (6) *converges absolutely.*

Proof. We have

$$\varphi(u) = u \int_0^\infty e^{-\frac{1}{2}ux} (ux)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(ux) f(x) \, dx, \tag{8}$$

and

$$\varphi(s) = \int_0^\infty (su)^{\frac{1}{2}} J_\nu(su) \varphi(u) \, du. \tag{9}$$

Using (8) in (9), we have

$$\varphi(s) = \int_0^\infty (su)^{\frac{1}{2}} J_\nu(su) \left[u \int_0^\infty e^{-\frac{1}{2}ux} (ux)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(ux) f(x) \, dx\right] du$$

$$= s^{\frac{1}{2}} \int_0^\infty x^{-1} f(x) \left[x \int_0^\infty e^{-\frac{1}{2}ux} (ux)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(ux) \, u^{3/2} J_\nu(su) \, du\right] dx.$$

Now, evaluating the u-integral by JAISWAL ([2], p. 389)

$$t^\mu J_\nu(at) \xrightarrow[m]{k+\frac{1}{2}} \frac{(a/2)^\nu \Gamma_*(\mu + \nu - k + 1 \pm m)}{s^{\mu+\nu} \Gamma(\mu + \nu - 2k + 1) \Gamma(\nu + 1)}.$$

$$\cdot {}_4F_3 \left\{ \begin{matrix} (\mu + \nu - k + 1 \pm m)/2, & (\mu + \nu - k + 2 \pm m)/2 \\ \nu + 1, \ (\mu + \nu - 2k + 1)/2, & (\mu + \nu - 2k + 2)/2 \end{matrix} ; -a^2/s^2 \right\},$$

$$\mathrm{Re}(\mu + \nu - k) + 1 > |\mathrm{Re}\, m|, \qquad \mathrm{Re}\, s > 0 \qquad \text{and} \qquad |s| > |a|,$$

we obtain the required result.

The change in the order of integration can be easily justified by DE LA VALLÉE POUSSIN's theorem ([5], p. 504) under the conditions stated in the Theorem.

3.1. - Particular cases.

(i) If in Theorem II we take $\nu = 1/2$, we get the following result (²): *If* $f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s)$ *and* $\varphi(s)$ *is* R_s^*, *then*

$$\varphi(s) = \sqrt{2/\pi}\, \frac{\Gamma_*(3-k\pm m)}{\Gamma(3-2k)}\, s \int_0^\infty x^{-3}\, {}_4F_3 \left\{ \begin{matrix} (3-k\pm m)/2,\ (4-k\pm m)/2 \\ 3/2,\ (3-2k)/2\ ,\ 2-2k \end{matrix} ; -s^2/x^2 \right\} f(x)\, dx$$

(*under the same conditions as of Theorem* II *with* $\nu = 1/2$).

(ii) On the other hand if we take $\nu = -1/2$ in Theorem II, we get the following result: *If* $f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s)$ *and* $\varphi(s)$ *is* R_c^*, *then*

$$\varphi(s) = \sqrt{2/\pi}\, \frac{\Gamma_*(2-k\pm m)}{\Gamma(2-2k)} \int_0^\infty x^{-2}\, {}_4F_3 \left\{ \begin{matrix} (2-k\pm m)/2,\ (3-k\pm m)/2 \\ 1/2,\ 1-k,\ (3-2k)/2 \end{matrix} ; -s^2/x^2 \right\} f(x)\, dx$$

(*under the same conditions as of Theorem* II *with* $\nu = -1/2$).

3.2. - Example.

Let $\varphi(s) = s^{-1/2}$. Then

$$f(t) = \frac{\Gamma((3/2)-2k)}{\Gamma_*((3/2)-k\pm m)}\, t^{1/2} \qquad \text{(JAISWAL [2], p. 387)},$$

provided $\mathrm{Re}((3/2)-k) > |\,\mathrm{Re}\, m\,|$, $\mathrm{Re}\, s > 0$.

It may be noted that $s^{-1/2}$ is the self-reciprocal in HANKEL transform of order ν ([6], p. 150).

Hence from Theorem II we get

$$\frac{\Gamma((3/2)-2k)}{\Gamma_*(\frac{1}{2}-k\pm m)} \int_0^\infty x^{1/2}\, G_\nu(s/x)\, dx = \frac{1}{\sqrt{s}}, \tag{10}$$

provided $\mathrm{Re}(\nu-k)+5/2 > |\,\mathrm{Re}\, m\,|$, $\mathrm{Re}\,((3/2)-k) > |\,\mathrm{Re}\, m\,|$, $\mathrm{Re}\, s > 0$ and $G_\nu(s/x)$ is given by (7).

(²) A function $f(t)$ is R_c^* if $f(t) = \sqrt{2/\pi}\int_0^\infty \cos(st)\, f(s)\, ds$, and is R_s^* if $f(t) = \sqrt{2/\pi}\,\cdot$ $\cdot \int_0^\infty \sin(st)\, f(s)\, ds$.

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