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Generalizzazioni dei polinomi di Laguerre e dei polinomi attuariali. (**)

1. - Introduzione.

In questo lavoro daremo un contributo allo sviluppo della teoria di due classi di polinomi più generali dei polinomi di LAGUERRE:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} D^n(x^{n+\alpha} e^{-x}) \qquad \left(D = D_x = \frac{d}{dx}\right), \tag{1.1}$$

e dei polinomi *attuariali*:

$$G_n^{(\alpha)}(x) = x^{-\alpha} e^x \underleftarrow{(xD)}^n (x^\alpha e^{-x}) . \tag{1.2}$$

I polinomi di LAGUERRE sono abbastanza noti. I polinomi *attuariali* sono stati considerati e posti in evidenza da J. F. STEFFENSEN in ricerche di matematica attuariale ([16], [17]), e noi, successivamente, ne abbiamo approfondito e ampliato il loro studio [23]. Alcuni dei nostri risultati sono stati riportati nel Vol. 3 dell'opera *Higher Transcendental Functions* del «Bateman Project» [10], dove si trova pure citata una monografia di C. TRUESDELL [27] a proposito di una equazione funzionale legata agli stessi polinomi. Ma questo non è sufficiente per attribuirli a C. TRUESDELL, come ha fatto recentemente R. P. SINGH in un suo lavoro di generalizzazione [14]. E noi continueremo a indicarli con il nome di polinomi *attuariali* già accettato da R. P. BOAS jr. e R. C. BUCK [2].

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A. ANGELESCO [1] ha brevemente considerato nel 1923 i polinomi più generali di quelli di LAGUERRE

$$x^{-\alpha} e^{x^p} \mathrm{D}^n (x^{n+\alpha} e^{-x^p}) ,$$

i quali per $p = 2$ e $\alpha = 0$ si riducono ad altri studiati da P. HUMBERT [12] nel 1923 e da M. DE DUFFAHEL [9] nel 1936.

H. M. SRIVASTAVA [15] ha studiato nella sua tesi di dottorato del 1954 i polinomi

$$u^n x^{-(\nu+n+1)/u} e^x (\underleftarrow{x^{1+(1/u)} \mathrm{D}})^n [x^{(\nu+1)/u} e^{-x}] ,$$

e qualche risultato si trova in una Nota del 1956 da lui pubblicata in collaborazione con A. SHARMA [13].

Il primo studio organico di polinomi più generali dei polinomi di LAGUERRE e dei polinomi *attuariali* risale al 1956 e si riferisce ai polinomi

$$x^{-\alpha} e^x x^{n(1-u)} (\underleftarrow{x^u \mathrm{D}})^n (x^\alpha e^{-x}) .$$

Esso è dovuto a A. M. CHAK [3], che ne trasse ispirazione dalla lettura del nostro lavoro [23] e che seguì pure nella sua trattazione.

H. W. GOULD e A. T. HOPPER [11] hanno studiato nel 1962 i polinomi

$$(-1)^n x^{-\alpha} e^{px^r} \mathrm{D}^n (x^\alpha e^{-px^r}) .$$

S. K. CHATTERJEA [4] ha studiato nel 1963 i polinomi

$$\frac{1}{n!} x^{-\alpha} e^{x^k} \mathrm{D}^n (x^{n+\alpha} e^{-x^k}) ,$$

e dal 1964, con impegno e fervore, i più generali ([5], [6], [7], [8])

$$\frac{1}{n!} x^{-\alpha} e^{px^k} \mathrm{D}^n (x^{n+\alpha} e^{-px^k}) .$$

Nel 1967 R. P. SINGH [14] ha studiato, come abbiamo accennato sopra, i polinomi

$$T_n^\alpha(x, r, p) = x^{-\alpha} e^{px^r} (\underleftarrow{x\mathrm{D}})^n (x^\alpha e^{-px^r}) .$$

Ma noi proveremo qui che si possono facilmente ricondurre ai polinomi *attuariali*.

Infine richiamiamo un nostro lavoro [24] del 1956 sui polinomi

$$L_n^{(\alpha,\nu)}(x) = \frac{(-1)^n \Gamma(\alpha + n\nu + 1)}{n!\, x^\alpha} \underset{\nu}{\Delta_\alpha^n} \frac{x^\alpha}{\Gamma(\alpha+1)}\,, \tag{1.3}$$

con $\underset{\nu}{\Delta_\alpha} f(\alpha) = f(\alpha+\nu) - f(\alpha)$. Essi estendono i polinomi di LAGUERRE considerati rispetto al parametro α e ad essi si riducono per $\nu = 1$.

I polinomi generalizzati richiamati, a parte gli $L_n^{(\alpha,\nu)}(x)$, sono chiaramente dipendenti tra di loro: per cui nel proseguimento di ricerche bisognerà stabilire il tipo da cui prendere le mosse.

Qui ci occuperemo dei polinomi

$$G_n^{(\alpha)}(x;\, u) = x^{-\alpha} e^x x^{n(1-u)} \big(\underleftarrow{x^u \mathrm{D}}\big)^n (x^\alpha e^{-x}) \tag{1.4}$$

che si riducono a quelli di LAGUERRE e agli *attuariali* secondo le relazioni

$$G^{(\alpha+n)}(x;\, 0) = G_n^{(\alpha+1)}(x;\, 2) = n!\; L_n^{(\alpha)}(x)\,, \tag{1.5}$$

$$G_n^{(\alpha)}(x;\, 1) = G_n^{(\alpha)}(x)\,, \tag{1.6}$$

a quelli di HERMITE e di BESSEL

$$H_n(x) = (-1)^n e^{x^2/2} \mathrm{D}^n e^{-x^2/2}\,,$$
$$y_n(x, a, b) = (1/b)^n x^{2-a} e^{b/x} \mathrm{D}^n (x^{a+2n-2} e^{-b/x})\,,$$

secondo le relazioni

$$G_n^{(0)}(x^2/2;\, \tfrac{1}{2}) = (-x/2)^n H_n(x)\,, \tag{1.7}$$

$$G_n^{(2-a-2n)}(b/x;\, 2) = (-b/x)^n\, y_n(x, a, b)\,. \tag{1.8}$$

Vanno pure segnalati i polinomi particolari:

$$G_n^{(\alpha)}(x;\, -1) = x^{2n-\alpha} e^x \big(\underleftarrow{x^{-1} \mathrm{D}}\big)^n (x^\alpha e^{-x}) = 2^n x^{2n-\alpha} e^x \mathrm{D}_{x^2}^n (x^\alpha e^{-x})\,, \tag{1.9}$$

$$G_n^{(\alpha)}(x;\, \tfrac{1}{2}) = x^{(n/2)-\alpha} e^x \big(\underleftarrow{x^{1/2} \mathrm{D}}\big)^n (x^\alpha e^{-x}) = (\tfrac{1}{2})^n x^{(n/2)-\alpha} e^x \mathrm{D}_{x^{1/2}}^n (x^\alpha e^{-x})\,, \tag{1.10}$$

$$G_n^{(\alpha)}(x;\, \tfrac{3}{2}) = x^{-(n/2)-\alpha} e^x \big(\underleftarrow{x^{3/2} \mathrm{D}}\big)^n (x^\alpha e^{-x}) = (-\tfrac{1}{2})^n x^{-(n/2)-\alpha} e^x \mathrm{D}_{x^{-1/2}}^n (x^\alpha e^{-x})\,, \tag{1.11}$$

$$G_n^{(\alpha)}(x;\, 3) = x^{-2n-\alpha} e^x \big(\underleftarrow{x^3 \mathrm{D}}\big)^n (x^\alpha e^{-x}) = (-2)^n x^{-2n-\alpha} e^x \mathrm{D}_{x^{-2}}^n (x^\alpha e^{-x})\,, \tag{1.12}$$

che in conseguenza della formula operatoria ([19], [26])

$$x^{n(1-u)} \big(\underleftarrow{x^u \mathrm{D}}\big)^n = x^{u-1} \underleftarrow{\big(\underleftarrow{x^{2-u} \mathrm{D}}\big)^n x^{(n-1)(u-1)}}$$

sono legati dalle formule:

$$G_n^{(\alpha)}(x;\,3) = G_n^{(\alpha+2n-2)}(x;\,-1)\,, \tag{1.13}$$

$$G_n^{(\alpha)}(x;\,\tfrac{3}{2}) = G_n^{[\alpha+(n-1)/2]}(x;\,\tfrac{1}{2})\,. \tag{1.14}$$

Il nostro contributo sui polinomi $G_n^{(\alpha)}(x;\,u)$ sarà fondato sull'applicazione, trascurata dagli Autori citati, dell'operazione di differenza finita rispetto al parametro α e sull'applicazione di un notevole gruppo di formule generali su gli operatori differenziali, a parametro u,

$$U^n\,\underleftarrow{(x^u\mathrm{D})}^n, \qquad \underleftarrow{\underleftarrow{(x^u\mathrm{D})}^n\,U^n}, \qquad U^n\,\underleftarrow{(\mathrm{D}x^u)}^n, \qquad \underleftarrow{\underleftarrow{(\mathrm{D}x^u)}^n\,U^n},$$

con $U = x^{1-u}$, e su gli analoghi [26] a parametro v, gruppo che per l'occasione verrà ampliato.

Infine, introdurremo e studieremo i nuovi polinomi

$$G_n^{(\alpha,\,\nu)}(x) = \frac{(-1)^n\,\Gamma[(\alpha/\nu)+1]}{\nu^n x^\alpha}\,\underleftarrow{(\Delta_{\underset{\nu}{\alpha}}\,\alpha)^n}\,\frac{x^\alpha}{\Gamma(\alpha+1)}\,, \tag{1.15}$$

che generalizzano gli *attuariali* ($\nu = 1$) come gli $L_n^{(\alpha,\,\nu)}(x)$ generalizzano quelli di LAGUERRE.

La generatrice degli $L_n^{(\alpha,\,\nu)}(x)$ ha messo in luce la funzione di MAITLAND WRIGHT [24], più generale di quella di BESSEL; quella dei $G_n^{(\alpha,\,\nu)}(x)$ metterà in luce una nuova generalizzazione della funzione di BESSEL, di cui si potrebbe fare un esame approfondito.

Il lavoro continua nostre precedenti ricerche con ulteriori sviluppi, approfondimenti, applicazioni. I risultati ottenuti, a parte qualcuno noto ma ritrovato con originalità di procedimento, credo siano nuovi.

2. - I polinomi di R. P. Singh.

Dapprima facciamo vedere che i polinomi generalizzati di R. P. SINGH non costituiscono una vera estensione e quindi non richiedono uno studio particolare, se non per il parametro p, in quanto essi e le loro proprietà si possono dedurre da quelle dei polinomi *attuariali* con una elementare sostituzione.

Osserviamo che

$$p\,r\,x^{r-1}\,\mathrm{D}_{p\,x^r} = \mathrm{D},$$

da cui

$$r^n \underleftarrow{(p\, x^r\, \mathrm{D}_{p\,x^r})^n} = \underleftarrow{(x\,\mathrm{D})^n}.$$

Risaliamo alla loro definizione e si ha

$$T_n^\alpha(x, r, p) = r^n\, x^{-\alpha}\, e^{p\,x^r} \underleftarrow{(p\, x^r\, \mathrm{D}_{p\,x^r})^n}\, (x^\alpha\, e^{-p\,x^r}) =$$

$$= r^n\, (p\, x^r)^{-\alpha/r}\, e^{p\,x^r} \underleftarrow{(p\, x^r\, \mathrm{D}_{p\,x^r})^n}\, [(p\, x^r)^{\alpha/r}\, e^{-p\,x^r}]\,.$$

Questa equivale alla semplice e notevole relazione

$$T_n^\alpha(x, r, p) = r^n\, G_n^{(\alpha/r)}(p\, x^r) \tag{2.1}$$

che riconduce i polinomi di R. P. SINGH a quelli *attuariali.*

3. - Sviluppi su operatori differenziali.

Prendiamo le mosse dagli sviluppi noti [**26**]:

$$U^n \underleftarrow{(x^u\,\mathrm{D})^n} = \sum_1^n{}_m\, \alpha_{n,m}^{(u)}\, x^m\, \mathrm{D}^m, \tag{3.1}$$

$$U^n \underleftarrow{(x^u\,\mathrm{D})^n} = \sum_0^n{}_m\, (-1)^{n-m}\, \beta_{n+1,\,m+1}^{(u)}\, \underleftarrow{\mathrm{D}^m\, x^m}\,, \tag{3.2}$$

$$\underleftarrow{\mathrm{D}^n\, x^n} = \sum_1^n{}_m\, (-1)^{n-m}\, a_{n,\,m}^{(u)}\, \underleftarrow{\underleftarrow{(\mathrm{D}\, x^u)}^m\, U^m}\,, \tag{3.3}$$

$$x^n\, \mathrm{D}^n = \sum_0^n{}_m\, b_{n+1,\,m+1}^{(u)}\, \underleftarrow{\underleftarrow{(\mathrm{D}\, x^u)}^m\, U^m}\,, \tag{3.4}$$

dove, posto

$$(x, d, n) = x\,(x + d)\,(x + 2\,d)\,\ldots\,[x + (n-1)\,d]\,,$$

$$(x, d, 0) = 1\,, \qquad (x, 1, n) = (x, n)\,,$$

è:

$$\alpha^{(u)}_{n,m} = \frac{1}{m!}\,\Delta^m_x(x,\,u-1,\,n)\,, \qquad x=0;$$

$$\beta^{(u)}_{n+1,\,m+1} = \frac{1}{m!}\,\Delta^m_x(x,\,1-u,\,n)\,, \qquad x=1;$$

$$a^{(u)}_{n,m} = \frac{(-1)^{n-m}}{(u-1)^m}\,\frac{1}{m!}\,\underset{u-1}{\Delta^m_x}(x,\,n)\,, \qquad x=0;$$

$$b^{(u)}_{n+1,\,m+1} = \frac{(-1)^{n-m}}{(1-u)^m}\,\frac{1}{m!}\,\underset{1-u}{\Delta^m_x}(x,\,n)\,, \qquad x=1.$$

Ai primi due sviluppi corrispondono i più generali, anch'essi noti,

$$x^{-\alpha}\,U^n\,\underleftarrow{\underleftarrow{(x^u\mathrm{D})}^n\,x^\alpha} = \sum_{m=0}^{n} k^{(\alpha;\,u)}_{n+1,\,m+1}\,x^m\,\mathrm{D}^m, \tag{3.5}$$

$$x^{-\alpha}\,U^n\,\underleftarrow{\underleftarrow{(x^u\mathrm{D})}^n\,x^\alpha} = \sum_{m=0}^{n} (-1)^{n-m}\,k^{(1-\alpha;\,2-u)}_{n+1,\,m+1}\,\underleftarrow{\mathrm{D}^m\,x^m}, \tag{3.6}$$

con

$$k^{(\alpha;\,u)}_{n+1,\,m+1} = \frac{1}{m!}\,\Delta^m_\alpha(\alpha,\,u-1,\,n)\,.$$

Per estendere gli altri due, procediamo come appresso. Allo sviluppo (3.3) associamo l'altro

$$\underleftarrow{\underleftarrow{(\mathrm{D}\,x^u)}^m\,U^m} = m!\sum_{i=0}^{m}\frac{(1-u)^{m-i}}{i!}\binom{\frac{\alpha-u}{u-1}+m}{m-i}\,x^{-\alpha}\,U^i\,\underleftarrow{\underleftarrow{(x^u\mathrm{D})}^i\,x^\alpha}$$

e si ha

$$\underleftarrow{\mathrm{D}^n\,x^n} = \sum_{i=0}^{n}\left[\frac{1}{i!}\,x^{-\alpha}\,U^i\,\underleftarrow{\underleftarrow{(x^u\mathrm{D})}^i\,x^\alpha}\cdot\sum_{m=i}^{n}(-1)^{n-m}\,m!\,(1-u)^{m-i}\,a^{(u)}_{n,m}\binom{\frac{\alpha-u}{u-1}+m}{m-i}\right].$$

D'altra parte

$$(1-\alpha,\,n) = \sum_{m=1}^{n}(-1)^{n-m}\,m!\,(1-u)^m\,a^{(u)}_{n,m}\binom{\frac{\alpha-u}{u-1}+m}{m}\,,$$

da cui

$$\sum_{i}^{n}{}_{m} (-1)^{n-m} m!\,(1-u)^{m-i} a_{n,m}^{(u)} \binom{\dfrac{\alpha-u}{u-1}+m}{m-i} =$$

$$= \frac{1}{(1-u)^i} \Delta^{i}_{(\alpha-u)/(u-1)} (1-\alpha,\, n) = \frac{1}{(1-u)^i} \underset{1-u}{\Delta^{i}_{1-\alpha}} (1-\alpha,\, n)\,.$$

Quindi

$$\underleftarrow{\mathrm{D}^n x^n} = \sum_{0}^{n}{}_{i} \frac{1}{(1-u)^i} \frac{1}{i!} \underset{1-u}{\Delta^{i}_{1-\alpha}} (1-\alpha,\, n)\, x^{-\alpha}\, U^i \underleftarrow{\underleftarrow{(x^u \mathrm{D})^i}}\, x^\alpha\,.$$

E posto

$$\frac{(-1)^n}{(1-u)^m} \frac{1}{m!} \underset{u-1}{\Delta^{m}_{\alpha}} (\alpha,\, n) = h_{n+1,m+1}^{(\alpha;\,u)}\,,$$

risulta

$$\underleftarrow{\mathrm{D}^n x^n} = \sum_{0}^{n}{}_{m} (-1)^{n-m} h_{n+1,m+1}^{(1-\alpha;\,2-u)} x^{-\alpha}\, U^m \underleftarrow{\underleftarrow{(x^u \mathrm{D})^m}}\, x^\alpha. \tag{3.7}$$

Anche per lo sviluppo (3.4) procediamo come nel caso precedente. D'altra parte

$$(-1)^n (\alpha,\, n) = \sum_{0}^{n}{}_{m} m!\,(1-u)^m b_{n+1,\,m+1}^{(u)} \binom{\dfrac{\alpha-u}{u-1}+m}{m}\,,$$

da cui

$$\sum_{i}^{n}{}_{m} m!\,(1-u)^{m-i} b_{n+1,\,m+1}^{(u)} \binom{\dfrac{\alpha-u}{u-1}+m}{m-i} =$$

$$= \frac{(-1)^n}{(1-u)^i} \Delta^{i}_{(\alpha-u)/(u-1)}(\alpha,\, n) = \frac{(-1)^n}{(1-u)^i} \underset{u-1}{\Delta^{i}_{\alpha}}(\alpha,\, n)\,.$$

Allora

$$x^n \mathrm{D}^n = \sum_{0}^{n}{}_{i} \left[\frac{1}{i!} x^{-\alpha}\, U^i \underleftarrow{\underleftarrow{(x^u \mathrm{D})^i}}\, x^\alpha \sum_{i}^{n}{}_{m} m!\,(1-u)^{m-i} b_{n+1,\,m+1}^{(u)} \binom{\dfrac{\alpha-u}{u-1}+m}{m-i} \right] =$$

$$= \sum_{0}^{n}{}_{i} \frac{(-1)^n}{(1-u)^i} \frac{1}{i!} \underset{u-1}{\Delta^{i}_{\alpha}}(\alpha,\, n) \cdot x^{-\alpha}\, U^i \underleftarrow{\underleftarrow{(x^u \mathrm{D})^i}}\, x^\alpha,$$

e infine

$$x^n \mathrm{D}^n = \sum_{m=0}^{n} h^{(\alpha;u)}_{n+1,\,m+1}\, x^{-\alpha}\, U^m \left(x^u \mathrm{D}\right)^m x^{\alpha}. \tag{3.8}$$

I coefficienti $h^{(\alpha;u)}_{n+1,\,m+1}$, $k^{(\alpha;u)}_{n+1,\,m+1}$ si possono calcolare con le formule

$$h^{(\alpha;u)}_{n+1,m+1} = \frac{(-1)^n}{(u-1)^m}\,\frac{1}{m!}\sum_{i=0}^{m} (-1)^i \binom{m}{i} \left[\alpha + i(u-1),\, n\right],$$

$$k^{(\alpha;u)}_{n+1,m+1} = \frac{1}{m!}\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (\alpha + i,\, u-1,\, n),$$

e scambiare secondo l'altra

$$h^{(\alpha;u)}_{n+1,\,m+1} = (1-u)^{n-m}\, k^{(\alpha/(u-1);\, u/(u-1))}_{n+1,\,m+1}.$$

Le relative relazioni di ricorrenza sono:

$$\begin{aligned}
&h^{(\alpha;u)}_{n,1} = (-1)^{n-1}(\alpha,\, n-1), \qquad n > 1,\\
&h^{(\alpha;u)}_{n,n} = 1,\\
&h^{(\alpha;u)}_{n,m} = h^{(\alpha;u)}_{n-1,\,m-1} - [\alpha + (m-1)(u-1) + n - 2]\, h^{(\alpha;u)}_{n-1,\,m}, \qquad 1 < m < n,\\
&h^{(\alpha;u)}_{n,0} = 0, \qquad h^{(\alpha;u)}_{n,m} = 0 \qquad \text{per } m > n;
\end{aligned}$$

$$\begin{aligned}
&k^{(\alpha;u)}_{n,1} = (\alpha,\, u-1,\, n-1), \qquad n > 1,\\
&k^{(\alpha;u)}_{n,n} = 1,\\
&k^{(\alpha;u)}_{n,m} = k^{(\alpha;u)}_{n-1,\,m-1} + [\alpha + (n-2)(u-1) + m - 1]\, k^{(\alpha;u)}_{n-1,\,m}, \qquad 1 < m < n,\\
&k^{(\alpha;u)}_{n,0} = 0, \qquad k^{(\alpha;u)}_{n,m} = 0 \qquad \text{per } m > n.
\end{aligned}$$

Per $u = 1$ si hanno i coefficienti [**21**]

$$h^{(\alpha;1)}_{n,m} = h^{(\alpha)}_{n,m}, \qquad k^{(\alpha;1)}_{n,m} = k^{(\alpha)}_{n,m}.$$

Riportiamo, perchè fondamentali nell'economia della struttura di questo lavoro, delle formule operatorie che discendono da altre più generali che abbiamo stabilito per operatori permutabili di secondo ordine [**26**].

Associamo all'operatore $U = x^{1-u}$ l'altro $V = x^{1-v}$, ai coefficienti

$$a^{(u)}_{n,m}, \qquad b^{(u)}_{n+1,\,m+1}, \qquad \alpha^{(u)}_{n,m}, \qquad \beta^{(u)}_{n+1,\,m+1},$$

gli altri

$$a^{(u)}_{n+1,\,m+1} = \frac{(-1)^{n-m}}{(u-1)^m}\,\frac{1}{m!}\,\underset{u-1}{\Delta^m_x}(x,\,n)\,, \qquad x = u;$$

$$c^{(u)}_{n+1,\,m+1} = \frac{(-1)^{n-m}}{(1-u)^m}\,\frac{1}{m!}\,\underset{1-u}{\Delta^m_x}(x,\,n)\,, \qquad x = 1-u;$$

$$\alpha^{(u)}_{n+1,\,m+1} = \frac{1}{m!}\,\Delta^m_x(x,\,u-1,\,n)\,, \qquad x = u;$$

$$\gamma^{(u)}_{n+1,\,m+1} = \frac{1}{m!}\,\Delta^m_x(x,\,1-u,\,n)\,, \qquad x = 1-u;$$

poniamo $\lambda = (u-v)/(u-1)$, $\mu = (u-v)/(1-v)$.

Valgono le formule:

$$(3.9)\quad \begin{cases} U^n\,(\underleftarrow{x^u\mathrm{D}})^n = \sum_1^n{}_m\,(1-u)^{n-m}\,a^{(\lambda)}_{n,m}\;V^m\,(\underleftarrow{x^v\mathrm{D}})^m = \\ \qquad = \sum_1^n{}_m\,(1-v)^{n-m}\,\alpha^{(\mu)}_{n,m}\;V^m\,(\underleftarrow{x^v\mathrm{D}})^m\,, \end{cases}$$

$$(3.10)\quad \begin{cases} U^n\,(\underleftarrow{x^u\mathrm{D}})^n = \sum_0^n{}_m\,(1-u)^{n-m}\,c^{(\lambda)}_{n+1,\,m+1}\,\underleftarrow{\underline{(x^v\mathrm{D})^m\,V^m}} = \\ \qquad = \sum_0^n{}_m\,(v-1)^{n-m}\,\beta^{(\mu)}_{n+1,\,m+1}\,\underleftarrow{\underline{(x^v\mathrm{D})^m\,V^m}}\,, \end{cases}$$

$$(3.11)\quad U^n\,(\underleftarrow{x^u\mathrm{D}})^n = \sum_0^n{}_m\,\frac{1}{(1-v)^m}\,\frac{1}{m!}\left[\Delta^m_x((1-v)x,\,u-1,\,n),\;x = \frac{v}{v-1}\right]V^m\,(\underleftarrow{\mathrm{D}\,x^v})^m\,,$$

$$(3.12)\quad \begin{cases} U^n\,(\underleftarrow{x^u\mathrm{D}})^n = \\ = \sum_0^n{}_m\,\frac{1}{(v-1)^m}\,\frac{1}{m!}\left[\Delta^m_x((v-1)\,x,\,u-1,\,n),\;x = \frac{1}{1-v}\right]\underleftarrow{\underline{(\mathrm{D}\,x^v)^m\,V^m}}\,, \end{cases}$$

$$(3.13)\quad \begin{cases} \underleftarrow{\underline{(x^u\mathrm{D})^n\,U^n}} = \sum_0^n{}_m\,(u-1)^{n-m}\,a^{(\lambda)}_{n+1,\,m+1}\,\underleftarrow{\underline{(x^v\mathrm{D})^m\,V^m}} = \\ \qquad = \sum_0^n{}_m\,(v-1)^{n-m}\,\alpha^{(\mu)}_{n+1,\,m+1}\,\underleftarrow{\underline{(x^v\mathrm{D})^m\,V^m}}\,, \end{cases}$$

$$(3.14)\quad \begin{cases} \underleftarrow{\underline{(x^u\mathrm{D})^n\,U^n}} = \sum_0^n{}_m\,(u-1)^{n-m}\,b^{(\lambda)}_{n+1,\,m+1}\,V^m\,(\underleftarrow{x^v\mathrm{D}})^m = \\ \qquad = \sum_0^n{}_m\,(1-v)^{n-m}\,\gamma^{(\mu)}_{n+1,\,m+1}\,V^m\,(\underleftarrow{x^v\mathrm{D}})^m\,, \end{cases}$$

$$(3.15)\quad \begin{cases} \underleftarrow{\underleftarrow{(x^u \mathrm{D})}^n\, U^n} = \\ = \sum_0^n {}_m \dfrac{1}{(v-1)^m}\, \dfrac{1}{m!} \left[\Delta_x^m\big((v-1)\,x,\ 1-u,\ n\big),\ x = \dfrac{u}{1-v} \right] \underleftarrow{\underleftarrow{(\mathrm{D}\, x^v)}^m\ V^m}, \end{cases}$$

$$(3.16)\quad \begin{cases} \underleftarrow{\underleftarrow{(x^u \mathrm{D})}^n\, U^n} = \\ = \sum_0^n {}_m \dfrac{1}{(1-v)^m}\, \dfrac{1}{m!} \left[\Delta_x^m\big((1-v)\,x,\ 1-u,\ n\big),\ x=1 - \dfrac{u}{1-v} \right] V^m \underleftarrow{(\mathrm{D}\, x^v)}^m, \end{cases}$$

$$(3.17)\quad \begin{cases} U^n \underleftarrow{(\mathrm{D}\, x^u)}^n = \sum_0^n {}_m\, (1-u)^{n-m}\, a^{(\lambda)}_{n+1,\ m+1}\, V^m \underleftarrow{(\mathrm{D}\, x^v)}^m = \\ \qquad = \sum_0^n {}_m\, (1-v)^{n-m}\, \alpha^{(\mu)}_{n+1,\ m+1}\, V^m \underleftarrow{(\mathrm{D}\, x^v)}^m, \end{cases}$$

$$(3.18)\quad \begin{cases} U^n \underleftarrow{(\mathrm{D}\, x^u)}^n = \sum_0^n {}_m\, (1-u)^{n-m}\, b^{(\lambda)}_{n+1,\ m+1}\, \underleftarrow{\underleftarrow{(\mathrm{D}\, x^v)}^m\, V^m} = \\ \qquad = \sum_0^n {}_m\, (v-1)^{n-m}\, \gamma^{(\mu)}_{n+1,\ m+1}\, \underleftarrow{\underleftarrow{(\mathrm{D}\, x^v)}^m\, V^m}, \end{cases}$$

$$(3.19)\quad \begin{cases} U^n \underleftarrow{(\mathrm{D}\, x^u)}^n = \\ = \sum_0^n {}_m \dfrac{1}{(1-v)^m}\, \dfrac{1}{m!} \left[\Delta_x^m\big((1-v)\,x,\ u-1,\ n\big),\ x = \dfrac{u}{1-v} \right] V^m \underleftarrow{(x^v \mathrm{D})}^m, \end{cases}$$

$$(3.20)\quad \begin{cases} U^n \underleftarrow{(\mathrm{D}\, x^u)}^n = \\ = \sum_0^n {}_m \dfrac{1}{(v-1)^m}\, \dfrac{1}{m!} \left[\Delta_x^m\big((v-1)\,x,\ u-1,\ n\big),\ x=1 - \dfrac{u}{1-v} \right] \underleftarrow{\underleftarrow{(x^v \mathrm{D})}^m\, V^m}, \end{cases}$$

$$(3.21)\quad \begin{cases} \underleftarrow{\underleftarrow{(\mathrm{D}\, x^u)}^n\, U^n} = \sum_1^n {}_m\, (u-1)^{n-m}\, a^{(\lambda)}_{n,m}\, \underleftarrow{\underleftarrow{(D\, x^v)}^m\, V^m} = \\ \qquad = \sum_1^n {}_m\, (v-1)^{n-m}\, \alpha^{(\mu)}_{n,m}\, \underleftarrow{\underleftarrow{(\mathrm{D}\, x^v)}^m\ V^m}, \end{cases}$$

$$(3.22)\quad \begin{cases} \underleftarrow{\underleftarrow{(\mathrm{D}\, x^u)}^n\, U^n} = \sum_0^n {}_m\, (u-1)^{n-m}\, c^{(\lambda)}_{n+1,\ m+1}\, V^m \underleftarrow{(\mathrm{D}\, x^v)}^m = \\ \qquad = \sum_0^n {}_m\, (1-v)^{n-m}\, \beta^{(\mu)}_{n+1,\ m+1}\, V^m \underleftarrow{(\mathrm{D}\, x^v)}^m, \end{cases}$$

$$(3.23)\quad \begin{cases} \underleftarrow{\underleftarrow{(\mathrm{D}\, x^u)}^n\, U^n} = \\ = \sum_0^n {}_m \dfrac{1}{(v-1)^m}\, \dfrac{1}{m!} \left[\Delta_x^m\big((v-1)\,x,\ 1-u,\ n\big),\ x = \dfrac{v}{v-1} \right] \underleftarrow{\underleftarrow{(x^v \mathrm{D})}^m\ V^m}, \end{cases}$$

$$
(3.24)\qquad \left\{\begin{aligned} &\underleftarrow{(\underleftarrow{\mathrm{D}\,x^u})^n\,U^n} = \\ &= \sum_0^n{}_m \frac{1}{(1-v)^m}\,\frac{1}{m!}\left[\Delta_x^m\big((1-v)x,\,1-u,\,n\big),\;x=\frac{1}{1-v}\right]V^m\,(\underleftarrow{x^v\mathrm{D}})^m . \end{aligned}\right.
$$

Assegniamo ora altre formule operatorie che gioveranno nel seguito. La prima è

$$
(3.25)\qquad n!\;x^{-\alpha}\,\underleftarrow{U^n\,(\underleftarrow{x^u\mathrm{D}})^n\,x^\alpha} = \sum_0^n{}_m A_{n+1,m+1}^{(\alpha;\,u)}\,x^{n-m+1}\,\underleftarrow{\mathrm{D}^n\,x^{m-1}},
$$

i cui coefficienti sono definiti dalle posizioni

$$
\begin{aligned}
&A_{1,1}^{(\alpha;\,u)} = 1\,,\\
&A_{n,1}^{(\alpha;\,u)} = (-\alpha,\,1-u,\,n-1)\,, \qquad n>1\,,\\
&A_{n,n}^{(\alpha;\,u)} = (\alpha+1,\,u-1,\,n-1)\,, \qquad n>1\,,\\
&A_{n,m}^{(\alpha;\,u)} = [\alpha+(n-2)u-m+3]\,A_{n-1,\,m-1}^{(\alpha;\,u)} - [\alpha+(n-2)(u-1)-m+1]\,A_{n-1,m}^{(\alpha;\,u)}\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1<m<n\,,
\end{aligned}
$$

$$
A_{n,0}^{(\alpha;\,u)} = 0\,, \qquad A_{n,m}^{(\alpha;\,u)} = 0 \qquad \text{per } m>n;
$$

si possono calcolare con la formula

$$
A_{n,m}^{(\alpha;\,u)} = \sum_0^{n-m}{}_i\,(-1)^{n-m-i}\binom{n}{n-m-i}(\alpha+i+1,\,u-1,\,n-i),
$$

e soddisfano all'altra

$$
A_{n,n-m+1}^{(-\alpha-1;\,2-u)} = A_{n,m}^{(\alpha;\,u)}\,.
$$

Per stabilire la (3.25), supposta vera per i primi valori di n, la moltiplichiamo a sinistra per

$$
(n+1)\,U^{n+1}\,\underleftarrow{(x^u\mathrm{D})\,U^{-n}\,x^\alpha} = (n+1)\,\underleftarrow{[x\mathrm{D}+n(u-1]\,x^\alpha}.
$$

Si ha

$$
(n+1)!\;U^{n+1}\,\underleftarrow{(\underleftarrow{x^u\mathrm{D}})^{n+1}}\,x^\alpha =
$$

$$
= (n+1)\sum_0^n{}_m A_{n+1,\,m+1}^{(\alpha;\,u)}\,\underleftarrow{[x\mathrm{D}+n(u-1)]\,x^{\alpha+n-m+1}\,\mathrm{D}^n\,x^{m-1}} =
$$

$$= (n+1) \sum_{0}^{n}{}_m A^{(\alpha;\,u)}_{n+1,\,m+1}\, x\, \underleftarrow{\mathrm{D}\, x^{\alpha+n-m+1} \mathrm{D}^n x^{m-1}} +$$

$$+ n(n+1)(u-1) \sum_{0}^{n}{}_m A^{(\alpha;\,u)}_{n+1,\,m+1}\, x^{\alpha+n-m+1} \underleftarrow{\mathrm{D}^n x^{m-1}} =$$

$$= (n+1) \sum_{0}^{n}{}_m A^{(\alpha;\,u)}_{n+1,\,m+1}\, x\, \underleftarrow{[x^{\alpha+n-m+1}\mathrm{D} + (\alpha+n-m+1)\, x^{\alpha+n-m}]\, \mathrm{D}^n x^{m-1}} +$$

$$+ n(n+1)(u-1) \sum_{0}^{n}{}_m A^{(\alpha;\,u)}_{n+1,\,m+1}\, x^{\alpha+n-m+1} \underleftarrow{\mathrm{D}^n x^{m-1}} =$$

$$= (n+1) \sum_{0}^{n}{}_m A^{(\alpha;\,u)}_{n+1,\,m+1}\, x^{\alpha+n-m+2} \underleftarrow{\mathrm{D}^{n+1} x^{m-1}} +$$

$$+ (n+1) \sum_{0}^{n}{}_m [\alpha + n(u-1) + n - m + 1]\, A^{(\alpha;\,u)}_{n+1,\,u+1}\, x^{\alpha+n-m+1} \underleftarrow{\mathrm{D}^n x^{m-1}},$$

$$(n+1)!\; x^{-\alpha}\, U^{n+1} \underleftarrow{(x^u \mathrm{D})^{n+1} x^{\alpha}} =$$

$$= \sum_{0}^{n}{}_m [-\alpha + n(1-u) + m]\, A^{(\alpha;\,u)}_{n+1,\,m+1}\, x^{n-m+2} \underleftarrow{\mathrm{D}^{n+1} x^{m-1}} + \sum_{0}^{n}{}_m (\alpha + nu - m + 1)\, A^{(\alpha;\,u)}_{n+1,\,m+1} \cdot$$

$$\cdot \underleftarrow{[x^{n-m+2}\mathrm{D}^{n-m+2} + (n+1)\, x^{n-m+1}\mathrm{D}^{n-m+1}]\, \mathrm{D}^{m-1} x^{m-1}}\;.$$

E poichè

$$\underleftarrow{[x^{n-m+2}\mathrm{D}^{n-m+2} + (n+1)\, x^{n-m+1}\mathrm{D}^{n-m+1}]\, \mathrm{D}^{m-1} x^{m-1}} = x^{n-m+1} \underleftarrow{\mathrm{D}^{n+1} x^{m}},$$

segue:

$$(n+1)!\; x^{-\alpha}\, U^{n+1} \underleftarrow{\underleftarrow{(x^u \mathrm{D})}^{n+1} x^{\alpha}} =$$

$$= \sum_{0}^{n}{}_m [-\alpha + n(1-u) + m]\, A^{(\alpha;\,u)}_{n+1,\,m+1}\, x^{n-m+2} \underleftarrow{\mathrm{D}^{n+1} x^{m-1}} +$$

$$+ \sum_{0}^{n}{}_m (\alpha + nu - m + 1)\, A^{(\alpha;\,u)}_{n+1,\,m+1}\, x^{n-m+1} \underleftarrow{\mathrm{D}^{n+1} x^{m}}.$$

Ammesso che i coefficienti A siano nulli quando il secondo indice è nullo o maggiore del primo, risulta

$$(n+1)!\; x^{-\alpha}\, U^{n+1} \underleftarrow{\underleftarrow{(x^u \mathrm{D})}^{n+1} x^{\alpha}} = \sum_{0}^{n+1}{}_m \{[-\alpha + n(1-u) + m]\, A^{(\alpha;\,u)}_{n+1,\,m+1} +$$

$$+ (\alpha + nu - m + 2)\, A^{(\alpha;\,u)}_{n+1,\,m}\}\, x^{n-m+2} \underleftarrow{\mathrm{D}^{n+1} x^{m-1}},$$

e questa stabilisce la (3.25) e la relazione di ricorrenza dei suoi coefficienti.

Se nella (3.25) sostituiamo α con $-\alpha-1$, u con $2-u$, osserviamo che [19]

$$U^{-n}\,\underleftarrow{(x^{2-u}\mathrm{D})^n} = x\,\underleftarrow{\underleftarrow{(\mathrm{D}\,x^u)}^n\,U^n\,x^{-1}}$$

e scambiamo il coefficiente $A^{(-\alpha-1;\,2-u)}_{n+1,\,m+1}$ con $A^{(\alpha;\,u)}_{n+1,\,n-m+1}$, si ottiene lo sviluppo

$$n!\; x^{\alpha}\,\underleftarrow{\underleftarrow{(\mathrm{D}\,x^u)}^n\,U^n\,x^{-\alpha}} = \sum_0^n{}_m\, A^{(\alpha;\,u)}_{n+1,\,m+1}\, x^{m-1}\,\underleftarrow{\mathrm{D}^n\,x^{n-m+1}}\,. \tag{3.26}$$

Sempre dalla (3.25), sostituendo α con $\alpha+n(1-u)$ si ha lo sviluppo

$$n!\; x^{-\alpha}\,\underleftarrow{\underleftarrow{(x^u\mathrm{D})}^n\,U^n\,x^{\alpha}} = \sum_0^n{}_m\, A^{[\alpha+n(1-u);\,u]}_{n+1,\,m+1}\, x^{n-m+1}\,\underleftarrow{\mathrm{D}^n\,x^{m-1}}\,. \tag{3.27}$$

Le formule stabilite sono pure valide per operatori permutabili di secondo ordine.

In particolare, i coefficienti

$$A^{(0;\,u)}_{n+1,\,m+1}\,,\qquad A^{[n(1-u)-1;\,u]}_{n+1,\,m+1}\,,\qquad A^{(\alpha;\,1)}_{n+1,\,m+1}$$

si identificano rispettivamente con gli altri

$$A^{(u)}_{n,m}\,,\qquad B^{(u)}_{n+1,\,m+1}\,,\qquad L^{(-\alpha)}_{n+1,\,m+1}\,,$$

introdotti e studiati in precedenti lavori insieme alle formule operatorie cui appartengono ([20], [21], [22]).

Uno dei procedimenti seguiti in [26] per stabilire le (3.9), ..., (3.24) di questo lavoro si può applicare ad alcune formule della Nota [20] per generalizzarle. E si ottengono le seguenti:

$$n!\;\, U^n\,\underleftarrow{(x^u\mathrm{D})^n} = \sum_1^n{}_m\, A^{(\mu)}_{n,m}\, V^{n-m+1}\,\underleftarrow{\underleftarrow{(x^v\mathrm{D})}^n\, V^{m-1}}\,, \tag{3.28}$$

$$n!\;\, \underleftarrow{\underleftarrow{(\mathrm{D}\,x^u)}^n\, U^n} = \sum_1^n{}_m\, A^{(\mu)}_{n,m}\, V^{m-1}\,\underleftarrow{\underleftarrow{(\mathrm{D}\,x^v)}^n\, V^{n-m+1}}\,, \tag{3.29}$$

$$n!\;\, \underleftarrow{\underleftarrow{(x^u\mathrm{D})}^n\, U^n} = \sum_0^n{}_m\, B^{(\mu)}_{n+1,\,m+1}\, V^{n-m}\,\underleftarrow{\underleftarrow{(x^v\mathrm{D})}^n\, V^{m}}\,, \tag{3.30}$$

$$n!\;\, U^n\,\underleftarrow{(\mathrm{D}\,x^u)^n} = \sum_0^n{}_m\, B^{(\mu)}_{n+1,\,m+1}\, V^{m}\,\underleftarrow{\underleftarrow{(\mathrm{D}\,x^v)}^n\, V^{n-m}}\,. \tag{3.31}$$

Aggiungiamo infine i seguenti sviluppi ([21], [25]):

$$(3.32)\qquad \underleftarrow{(x\mathrm{D})}^{2n-1} = \sum_{1}^{n}{}_{m} (1-u)^{2n-2m} \underset{2}{k}_{n,m}\, \underleftarrow{U^{m} \underleftarrow{(x^{u}\mathrm{D})}^{2m-1}\, U^{m-1}} ,$$

$$(3.33)\qquad \underleftarrow{(x\mathrm{D})}^{2n} = \sum_{0}^{n}{}_{m} \left(\frac{1-u}{2}\right)^{2n-2m} T_{n+1,\,m+1}\, \underleftarrow{U^{m+(1/2)} \underleftarrow{(x^{u}\mathrm{D})}^{2m}\, U^{m-(1/2)}} ,$$

$$(3.34)\qquad 2^{n}\, n!\, \underleftarrow{(x\mathrm{D})}^{n} = \sum_{0}^{n}{}_{m} H_{n+1,\,m+1}\, \underleftarrow{U^{n-m+(1/2)} \underleftarrow{(x^{u}\mathrm{D})}^{n}\, U^{m-(1/2)}} ,$$

dove:

$$\underset{2}{k}_{n,1} = \underset{2}{k}_{n,n} = 1 ,$$

$$\underset{2}{k}_{n,m} = \underset{2}{k}_{n-1,\,m-1} + m^{2} \underset{2}{k}_{n-1,\,m} \qquad \text{per } 1 < m < n ,$$

$$\underset{2}{k}_{n,0} = 0 , \qquad \underset{2}{k}_{n,m} = 0 \qquad \text{per } m > n ,$$

$$\underset{2}{k}_{n,m} = (-1)^{m}\, 2 \sum_{1}^{m}{}_{i} \frac{(-1)^{i}\, i^{2n}}{(m-i)!\, (m+i)!} ;$$

$$T_{n,1} = T_{n,n} = 1 ,$$

$$T_{n,m} = T_{n-1,\,m-1} + (2m-1)^{2}\, T_{n-1,\,m} \qquad \text{per } 1 < m < n ,$$

$$T_{n,0} = 0 , \qquad T_{n,m} = 0 \qquad \text{per } m > n ,$$

$$T_{n+1,\,m+1} = \frac{1}{2^{2m}} \sum_{1}^{m+1}{}_{i} \frac{(-1)^{m-i+1}\, (2i-1)^{2n+1}}{(m-i+1)!\, (m+i)!} ;$$

$$H_{n,1} = H_{n,n} = 1 ,$$

$$H_{n,m} = (2n-2m+1)\, H_{n-1,\,m-1} + (2m-1)\, H_{n-1,\,m} \qquad \text{per } 1 < m < n ,$$

$$H_{n,0} = 0 , \qquad H_{n,m} = 0 \qquad \text{per } m > n ,$$

$$H_{n,n-m+1} = H_{n,m} ,$$

$$H_{n+1,\,m+1} = \sum_{0}^{m}{}_{i} (-1)^{i} \binom{n+1}{i} (2m-2i+1)^{n}.$$

Completiamo questo Capitolo, sulle formule operatorie differenziali, riportando dei valori particolari dei coefficienti che figurano in esse:

$$a_{n,m}^{(1)} = b_{n,m}^{(1)} = c_{n+1,\,m+1}^{(1)} = h_{n,m}\,,$$

$$\alpha_{n,m}^{(1)} = \beta_{n,m}^{(1)} = \gamma_{n+1,\,m+1}^{(1)} = k_{n,m}\,,$$

con $h_{n,m}$ e $k_{n,m}$ numeri di STIRLING di prima e seconda specie;

$$a_{n,m}^{(0)} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{per } m \begin{cases} < n \\ = n \end{cases},$$

$$a_{n,m}^{(2)} = (-1)^{n-m} \frac{n!}{m!} \binom{n-1}{m-1},$$

$$a_{n,m}^{(-1)} = \frac{1}{2^{n-m}} \frac{n!}{m!} \binom{m}{n-m},$$

$$a_{n,m}^{(1/2)} = \frac{(-1)^{n-m}}{2^{2n-2m}} \frac{(2n-m-1)!}{(n-1)!} \binom{n-1}{m-1},$$

$$b_{n,m}^{(0)} = (-1)^{n-m} \frac{(n-1)!}{(m-1)!} \binom{n-1}{m-1},$$

$$b_{n,m}^{(2)} = (-1)^{n-m} \frac{(n-1)!}{(m-1)!},$$

$$c_{n,m}^{(0)} = b_{n,m}^{(0)}, \qquad c_{n,m}^{(2)} = \begin{cases} 0 \\ n-1 \\ 1 \end{cases} \quad \text{per } m \begin{cases} < n-1 \\ = n-1 \\ = n \end{cases},$$

$$c_{n,m}^{(3)} = \frac{1}{2^{n-m}} \frac{(n-1)!}{(m+1)!} \binom{m+1}{n-m} [n(n-1) + 2m],$$

$$c_{n,m}^{(3/2)} = \frac{(-1)^{n-m}}{2^{2n-2m}} \frac{(2n-m-3)!}{(n-1)!} \binom{n-1}{m-1} [m(m-1) - 2n + 2],$$

$$\alpha_{n,m}^{(0)} = a_{n,m}^{(0)}, \qquad \alpha_{n,m}^{(2)} = \frac{n!}{m!} \binom{n-1}{m-1},$$

$$\alpha_{n,m}^{(-1)} = \frac{(-1)^{n-m}}{2^{n-m}} \frac{(2n-m-1)!}{(n-1)!} \binom{n-1}{m-1},$$

$$\alpha^{(1/2)}_{n,m} = \frac{1}{2^{2n-2m}} \frac{n!}{m!} \binom{m}{n-m} ,$$

$$\beta^{(0)}_{n,m} = \frac{(n-1)!}{(m-1)!} \binom{n-1}{m-1} , \qquad \beta^{(2)}_{n,m} = c^{(2)}_{n,m} ,$$

$$\beta^{(3/2)}_{n,m} = \frac{1}{2^{2n-2m}} \frac{(n-1)!}{(m+1)!} \binom{m+1}{n-m} [n(n-1) + 2m] ,$$

$$\beta^{(3)}_{n,m} = \frac{(-1)^{n-m}}{2^{n-m}} \frac{(2n-m-3)!}{(n-1)!} \binom{n-1}{m-1} [m(m-1) - 2n + 2] ,$$

$$\gamma^{(0)}_{n,m} = \beta^{(0)}_{n,m} , \qquad \gamma^{(2)}_{n,m} = b^{(2)}_{n,m} ;$$

$$A^{(1)}_{n,m} = B^{(1)}_{n+1,m} = A_{n,m} ,$$

con $A_{n,m}$ numeri di EULERO (1755);

$$A^{(2)}_{n,m} = \begin{cases} 0 \\ n! \end{cases} \quad \text{ner } m \begin{cases} < n \\ = n \end{cases} ,$$

$$A^{(1/2)}_{n,m} = \frac{n!}{2^n} \binom{n+1}{2m-1} , \qquad A^{(3/2)}_{n,m} = \frac{n!}{2^n} \binom{n+1}{2n-2m+1} ,$$

$$B^{(2)}_{n,m} = (-1)^{m+1} (n-1)! \binom{n}{m} , \qquad B^{(1/2)}_{n,m} = \frac{(n-1)!}{2^{n-1}} \binom{n}{2n-2m} ,$$

$$B^{(3/2)}_{n,m} = \frac{(n-1)!}{2^{n-1}} \left[\binom{n}{2m} + (-1)^{m-1} \binom{n}{m} \right] .$$

4. - I polinomi $G^{(\alpha)}_n(x; u)$.

Dalla relazione (1.4) che li definisce si ha ancora:

$$x^{-\alpha} e^{x} \underleftarrow{(x^u \mathrm{D})^n} \, U^n (x^\alpha e^{-x}) = G^{[\alpha+n(1-u)]}_n (x; u) ,$$

$$x^{-\alpha} e^{x} U^n \underleftarrow{(\mathrm{D}\, x^u)^n} (x^\alpha e^{-x}) = G^{(\alpha+u)}_n (x; u) ,$$

$$x^{-\alpha} e^{x} \underleftarrow{(\mathrm{D}\, x^u)^n \, U^n} (x^\alpha e^{-x}) = G^{[\alpha+u+n(1-u)]}_n (x; u) ,$$

$$x^{-\alpha}\, e^{x}\, U^{n-i}\, \underleftarrow{(\underleftarrow{x^{u}\mathrm{D}})^{n}}\, U^{i}\, (x^{\alpha}\, e^{-x}) = G_{n}^{[\alpha+i(1-u)]}(x;\, u)\,,$$

$$x^{-\alpha}\, e^{x}\, U^{n-i}\, \underleftarrow{(\underleftarrow{\mathrm{D}\, x^{u}})^{n}}\, U^{i}\, (x^{\alpha}\, e^{-x}) = G_{n}^{[\alpha+\ +i(1-u)]}(x;\, u)\,,$$

$$x^{-\alpha}\, e^{x}\, U^{i+(1/2)}\, \underleftarrow{(\underleftarrow{x^{u}\mathrm{D}})^{2i}}\, U^{i-(1/2)}\, (x^{\alpha}\, e^{-x}) = G_{2i}^{[\alpha+(i-\frac{1}{2})(1-u)]}(x;\, u)\,.$$

Applicando la (3.5) alla funzione e^{-x} si ha subito

$$G_{n}^{(\alpha)}(x;\, u) = \sum_{0}^{n}{}_{m}\, k_{n+1,\, m+1}^{(\alpha;\, u)}\, (-x)^{m}, \tag{4.1}$$

e i polinomi risultano di grado n. Mentre dalla (3.8) si ha l'inversa

$$(-x)^{n} = \sum_{0}^{n}{}_{m}\, h_{n+1,\, m+1}^{(\alpha;\, u)}\, G_{m}^{(\alpha)}(x;\, u)\,. \tag{4.2}$$

La (4.1) si può scrivere, in termini di differenze finite, nella forma

$$G_{n}^{(\alpha)}(x;\, u) = \sum_{0}^{n}{}_{m}\, \frac{(-x)^{m}}{m!}\, \Delta_{\alpha}^{m}(\alpha,\, u-1,\, n) = \left[\sum_{0}^{\infty}{}_{m}\, \frac{(-x\Delta_{\alpha})^{m}}{m!}\right](\alpha,\, u-1,\, n)\,,$$

da cui segue [3]

$$G_{n}^{(\alpha)}(x;\, u) = e^{-x\Delta\alpha}\, (\alpha,\, u-1,\, n)\,. \tag{4.3}$$

Da questa si ha

$$\underset{u-1}{\Delta_{\alpha}}\, G_{n}^{(\alpha)}(x;\, u) = \underset{u-1}{\Delta_{\alpha}}\, e^{-x\Delta\alpha}\, (\alpha,\, u-1,\, n) =$$

$$= e^{-x\Delta\alpha}\, \underset{u-1}{\Delta_{\alpha}}\, (\alpha,\, u-1,\, n) = n\, (u-1)\, e^{-x\Delta\alpha}\, (\alpha+u-1,\, u-1,\, n-1)\,,$$

e quindi

$$\underset{u-1}{\Delta_{\alpha}}\, G_{n}^{(\alpha)}(x;\, u) = n\, (u-1)\, G_{n-1}^{(\alpha+u-1)}(x;\, u)\,. \tag{4.4}$$

Per l'iterata m-esima viene

$$\underset{u-1}{\Delta_{\alpha}^{m}}\, G_{n}^{(\alpha)}(x;\, u) = \binom{n}{m} m!\, (u-1)^{m}\, G_{n-m}^{[\alpha+m(u-1)]}(x;\, u)\,. \tag{4.5}$$

D'altra parte per le differenze finite si ha

$$\underset{1-u}{\Delta_\alpha^m} = (-1)^m \underset{1-u}{\Delta_{-\alpha}^m} E_\alpha^{m(1-u)} = (-1)^m \underset{u-1}{\Delta_\alpha^m} E_\alpha^{m(1-u)},$$

con E_α operatore di *spostamento* definito dalla $E_\alpha f(\alpha) = f(\alpha+1)$, per cui $\Delta_\alpha = E_\alpha - 1$. Pertanto

$$\underset{1-u}{\Delta_\alpha^m} G_n^{(\alpha)}(x;\, u) = (-1)^m \underset{u-1}{\Delta_\alpha^m} E_\alpha^{m(1-u)} G_n^{(\alpha)}(x;\, u) =$$

$$= (-1)^m \underset{u-1}{\Delta_\alpha^m} G_n^{[\alpha+m(1-u)]}(x;\, u) =$$

$$= (-1)^m \binom{n}{m} m!\, (u-1)^m G_{n-m}^{[\alpha+m(1-u)+m(u-1)]}(x;\, u),$$

cioè

$$\underset{1-u}{\Delta_\alpha^m} G_n^{(\alpha)}(x;\, u) = \binom{n}{m} m!\, (1-u)^m G_{n-m}^{(\alpha)}(x;\, u). \tag{4.6}$$

Dalle differenze finite si possono ottenere le derivate dei polinomi G rispetto al parametro α, con l'applicazione delle formule:

$$(u-1)^m D_\alpha^m = \sum_m^\infty {}_i \frac{m!}{i!} h_{i,m} \underset{u-1}{\Delta_\alpha^i}, \qquad (1-u)^m D_\alpha^m = \sum_m^\infty {}_i \frac{m!}{i!} h_{i,m} \underset{1-u}{\Delta_\alpha^i}.$$

E si ha

$$D_\alpha^m G_n^{(\alpha)}(x;\, u) = \sum_m^n {}_i\, m! \binom{n}{i} h_{i,m} (u-1)^{i-m} G_{n-i}^{[\alpha+i(u-1)]}(x;\, u), \tag{4.7}$$

$$D_\alpha^m G_n^{(\alpha)}(x;\, u) = \sum_m^n {}_i\, m! \binom{n}{i} h_{i,m} (1-u)^{i-m} G_{n-i}^{(\alpha)}(x;\, u). \tag{4.8}$$

Dalla relazione (1.4) deduciamo

$$G_n^{(\alpha)}(x;\, u) = x^{-\alpha} e^x U^n \underbrace{(x^u D)}^n \sum_0^\infty {}_m \frac{(-1)^m}{m!} x^{\alpha+m},$$

ed è poi

$$x^{-\alpha}\, U^n\, \underleftarrow{(x^u \mathrm{D})}^n\, x^{\alpha+m} = (\alpha + m,\ u - 1,\ n)\, x^m\,.$$

Quindi

$$G_n^{(\alpha)}(x;\, u) = e^x \sum_0^\infty{}_m \frac{(-1)^m}{m!}\, (\alpha + m,\ u - 1, n)\, x^m\,,$$

da cui

$$\sum_0^\infty{}_m\, (\alpha + m,\ u - 1,\ n)\, \frac{(-x)^m}{m!} = e^{-x}\, G_n^{(\alpha)}(x;\, u)\,. \tag{4.9}$$

I risultati (4.3), (4.9) ci serviranno ora per stabilire le due principali funzioni generatrici dei polinomi G per via elegante e diversa da quella seguita dagli Autori citati. Si ha:

$$\sum_0^\infty{}_n \frac{t^n}{n!}\, G_{n+m}^{(\alpha)}(x;\, u) = \sum_0^\infty{}_n \frac{t^n}{n!}\, e^{-x\,\Delta_\alpha}(\alpha,\ u - 1,\ n + m) =$$

$$= e^x e^{-x\,\mathrm{E}_\alpha}(\alpha,\ u - 1,\ m) \sum_0^\infty{}_n \frac{[t(u-1)]^n}{n!} \left(\frac{\alpha}{u-1} + m,\ n\right) =$$

$$= [1 + t\,(1 - u)]^{-m}\, e^x e^{-x\,\mathrm{E}_\alpha}\big[(\alpha,\ u - 1,\ m)(1 + t\,(1 - u))^{\alpha/(1-u)}\big] =$$

$$= [1 + t\,(1 - u)]^{-m}\, e^x \sum_0^\infty{}_n \frac{(-x)^n}{n!}\, (\alpha + n,\ u - 1,\ m)[1 + t\,(1 - u)]^{(\alpha+n)/(1-u)} =$$

$$= [1 + t\,(1 - u)]^{-m+\alpha/(1-u)}\, e^x \sum_0^\infty{}_n \frac{(\alpha + n,\ u - 1,\ m)}{n!}\, [-x\,(1 + t\,(1 - u))^{1/(1-u)}]^n,$$

da cui, per la (4.9),

$$\left\{\begin{aligned} &\sum_0^\infty{}_n \frac{t^n}{n!}\, G_{n+m}^{(\alpha)}(x;\, u) = \\ &= [1 + t\,(1 - u)]^{-m+\alpha/(1-u)}\, \exp x\big[1 - (1 + t\,(1 - u))^{1/(1-u)}\big]\,\cdot \\ &\qquad \cdot\, G_m^{(\alpha)}\big[x\,(1 + t\,(1 - u))^{1/(1-u)};\, u\big]\,. \end{aligned}\right. \tag{4.10}$$

Dalla identità

$$(\alpha + n\,(1 - u),\ u - 1,\ n + m) = (\alpha,\ u - 1,\ m)(\alpha + 1 - u,\ 1 - u,\ n)$$

viene come sopra

$$\sum_0^\infty{}_n \frac{t^n}{n!} G_{n+m}^{[\alpha+n(1-u)]}(x;\, u) =$$

$$= e^x e^{-x\,\mathrm{E}_\alpha}(\alpha,\, u-1,\, m) \sum_0^\infty{}_n \frac{[t(1-u)]^n}{n!} \left(\frac{\alpha}{1-u}+1,\, n\right) =$$

$$= e^x e^{-x\,\mathrm{E}_\alpha}\left[(\alpha,\, u-1,\, m)(1+t(u-1))^{\{\alpha/(u-1)\}-1}\right] =$$

$$= e^x \sum_0^\infty{}_n \frac{(-x)^n}{n!} (\alpha+n,\, u-1,\, m)[1+t(u-1)]^{\{(\alpha+n)/(u-1)\}-1} =$$

$$= [1+t(u-1)]^{\{\alpha/(u-1)\}-1} e^x \sum_0^\infty{}_n \frac{(\alpha+n,\, u-1,\, m)}{n!} \left[-x(1+t(u-1))^{1/(u-1)}\right]^n ,$$

e a conclusione

$$(4.11)\quad \left\{\begin{aligned} &\sum_0^\infty{}_n \frac{t^n}{n!} G_{n+m}^{[\alpha+n(1-u)]}(x;\, u) = \\ &= [1+t(u-1)]^{\{\alpha/(u-1)\}-1} \exp x\left[1-(1+t(u-1))^{1/(u-1)}\right] \cdot \\ &\qquad \cdot G_m^{(\alpha)}\left[x(1+t(u-1))^{1/(u-1)};\, u\right] . \end{aligned}\right.$$

Supponiamo ora $m = 0$ e consideriamo i due sviluppi

$$(4.10')\quad \left\{\begin{aligned} &\sum_0^\infty{}_n \frac{t^n}{n!} G_n^{(\alpha)}(x;\, u) = \\ &= [1+t(1-u)]^{\alpha/(1-u)} \exp x\left[1-(1+t(1-u))^{1/(1-u)}\right] , \end{aligned}\right.$$

$$(4.11')\quad \left\{\begin{aligned} &\sum_0^\infty{}_n \frac{t^n}{n!} G_n^{[\alpha+n(1-u)]}(x;\, u) = \\ &= [1+t(u-1)]^{\{\alpha/(u-1)\}-1} \exp x\left[1-(1+t(u-1))^{1/(u-1)}\right] . \end{aligned}\right.$$

Da essi seguono facilmente le note formule di addizione

$$(4.12)\quad \left\{\begin{aligned} &G_n^{(\alpha_1+\alpha_2+\ldots+\alpha_m)}(x_1+x_2+\ldots+x_m;\, u) = \\ &= \sum \frac{n!}{i_1!\, i_2!\, \ldots\, i_m!} G_{i_1}^{(\alpha_1)}(x_1;\, u) \ldots G_{i_m}^{(\alpha_m)}(x_m;\, u) , \end{aligned}\right.$$

$$(4.13)\quad \left\{\begin{aligned} &G_n^{[\alpha_1+\alpha_2+\ldots+\alpha_m+(n+m-1)(1-u)]}(x_1+x_2+\ldots+x_m;\, u) = \\ &= \sum \frac{n!}{i_1!\, i_2!\, \ldots\, i_m!} G_{i_1}^{[\alpha_1+i_1(1-u)]}(x_1;\, u) \ldots G_{i_m}^{[\alpha_m+i_m(1-u)]}(x_m;\, u) , \end{aligned}\right.$$

con $i_1, i_2, \ldots, i_m$ interi positivi o nulli, tali che $i_1+i_2+\ldots+i_m = n$.

Se deriviamo rispetto a t i due membri dello sviluppo (4.10′), il cui secondo denotiamo brevemente con φ, si ha

$$[1+t(1-u)]^{u/(u-1)}\,\frac{\partial\varphi}{\partial t}=[\alpha\,(1+t(1-u))^{1/(u-1)}-x]\,\varphi\,.$$

Sviluppando viene:

$$\sum_0^\infty{}_i\,\frac{(-t)^i}{i!}\,(u,\,1-u,\,i)\,\sum_0^\infty{}_j\,\frac{t^j}{j!}\,G^{(\alpha)}_{j+1}(x;\,u)=$$

$$=\sum_0^\infty{}_r\,\frac{t^r}{r!}\,G^{(\alpha)}_r(x;\,u)\left[\alpha\sum_0^\infty{}_s\,\frac{(-t)^s}{s!}\,(1,\,1-u,\,s)-x\right],$$

e uguagliando i coefficienti di t^n:

$$\sum_0^n{}_i\,(-1)^{n-i}\binom{n}{i}\,(u,\,1-u,\,n-i)\,G^{(\alpha)}_{i+1}(x;\,u)-$$

$$-\alpha\sum_0^n{}_i\,(-1)^{n-i}\binom{n}{i}\,(1,\,1-u,\,n-i)\,G^{(\alpha)}_i(x;\,u)+x\,G^{(\alpha)}_n(x;\,u)=0\,.$$

Quindi risulta la relazione di ricorrenza $(n\geqslant 2)$

$$(4.14)\quad\left\{\begin{aligned}&G^{(\alpha)}_{n+1}(x;\,u)+(x-\alpha-nu)\,G^{(\alpha)}_n(x;\,u)+\\&+\sum_0^{n-2}{}_i\,(-1)^{n-i}\left[\binom{n}{i}(u,\,1-u,\,n-i)+\alpha\binom{n}{i+1}(1,\,1-u,\,n-i-1)\right]\cdot\\&\cdot G^{(\alpha)}_{i+1}(x;\,u)+(-1)^{n+1}\alpha\,(1,\,1-u,\,n)=0\,.\end{aligned}\right.$$

Analogamente dalla (4.11′) segue l'altra relazione di ricorrenza $(n\geqslant 2)$

$$(4.15)\quad\left\{\begin{aligned}&G^{[\alpha+(n+1)(1-u)]}_{n+1}(x;\,u)+[x-\alpha-n+(n+1)(u-1)]\,G^{[\alpha+n(1-u)]}_n(x;\,u)+\\&+\sum_0^{n-2}{}_i\,(-1)^{n-i}\left[\binom{n}{i}(2-u,\,u-1,\,n-i)+\right.\\&\left.+(\alpha-u+1)\binom{n}{i+1}(1,\,u-1,\,n-i-1)\right]G^{[\alpha+(i+1)(1-u)]}_{i+1}(x;\,u)+\\&\qquad\qquad+(-1)^{n+1}(\alpha-u+1)\,(1,\,u-1,\,n)=0\,.\end{aligned}\right.$$

Ancora due relazioni ricorrenti si possono ottenere dalle formule operatorie:

$$U^n\,(\underleftarrow{x^u\mathrm{D}})^n=U^{n-1}\,\underleftarrow{(x^u\mathrm{D})^{n-1}(x\mathrm{D})}+(n-1)(u-1)\,U^{n-1}\,(\underleftarrow{x^u\mathrm{D}})^{n-1}\,,$$

$$\underleftarrow{(x^u\mathrm{D})^n\,U^n}=\underleftarrow{(x^u\mathrm{D})^{n-1}\,U^{n-1}\,(x\mathrm{D})}+n(1-u)\underleftarrow{(x^u\mathrm{D})^{n-1}\,U^{n-1}}\,,$$

e sono ($n \geq 1$)

$$G_n^{(\alpha)}(x;\, u) = [\alpha + (n-1)(u-1)]\, G_{n-1}^{(\alpha)}(x;\, u) - x\, G_{n-1}^{(\alpha+1)}(x;\, u)\,, \tag{4.16}$$

$$G_n^{(\alpha-u+1)}(x;\, u) = (\alpha - u + 1)\, G_{n-1}^{(\alpha)}(x;\, u) - x\, G_{n-1}^{(\alpha+1)}(x;\, u)\,. \tag{4.17}$$

A questo punto giova osservare che vale la formula

$$G_n^{(\alpha)}(x;\, u) = G_n^{[\alpha+(n-1)(u-1)]}(x;\, 2-u)\,, \tag{4.18}$$

e quindi che (esclusi i casi $u = 0$, $u = 2$) per essa si può passare direttamente dalle (4.10′), (4.12), (4.14) alle (4.11′), (4.13), (4.15).

Dagli sviluppi (4.10′) e (4.11′), per derivazione rispetto a x, si ottengono le

$$\mathrm{D}_x G_n^{(\alpha)}(x;\, u) = -\Delta_\alpha G_n^{(\alpha)}(x;\, u)\,, \tag{4.19}$$

$$\mathrm{D}_x G_n^{[\alpha+n(1-u)]}(x;\, u) = -\Delta_\alpha G_n^{[\alpha+n(1-u)]}(x;\, u)\,. \tag{4.19′}$$

D'altra parte dalla formula

$$G_n^{(\beta)}(x;\, u) = \sum_0^n{}_m \binom{n}{m} (\beta - \alpha,\, u-1,\, n-m)\, G_m^{(\alpha)}(x;\, u)\,, \tag{4.20}$$

derivata da quella di addizione (4.12), per $\beta = \alpha + 1$ viene

$$\sum_0^{n-1}{}_m \binom{n}{m} (1,\, u-1,\, n-m)\, G_m^{(\alpha)}(x;\, u) = G_n^{(\alpha+1)}(x;\, u) - G_n^{(\alpha)}(x;\, u) = \Delta_\alpha\, G_n^{(\alpha)}(x;\, u)\,.$$

Pertanto dalla (4.19) si ha ($n \geq 1$)

$$\mathrm{D}_x G_n^{(\alpha)}(x;\, u) = -\sum_0^{n-1}{}_m \binom{n}{m} (1,\, u-1,\, n-m)\, G_m^{(\alpha)}(x;\, u)\,. \tag{4.21}$$

Dalla seconda formula di addizione (4.13), con $\alpha_1 = \alpha + 1$, $\alpha_2 = u - 2$, $x_1 = x$, $x_2 = 0$, viene

$$G_n^{[\alpha+n(1-u)]}(x;\, u) = \sum_0^n{}_m (-1)^{n-m} (1,\, u-1,\, n-m)\, G_m^{[\alpha+m(1-u)+1]}(x;\, u)\,.$$

E per la (4.19′) si ottiene ($n \geq 1$):

$$\mathrm{D}_x G_n^{[\alpha+n(1-u)]}(x;\, u) = \sum_0^{n-1}{}_m (-1)^{n-m} \binom{n}{m} (1,\, u-1,\, n-m)\, G_m^{[\alpha+m(1-u)+1]}(x;\, u)\,. \tag{4.22}$$

La formula (4.19) ci consentirà ora di pervenire rapidamente ad un nuovo sviluppo in serie.

Dallo sviluppo elementare

$$\sum_0^\infty {}_i \frac{t^i}{i!} \mathrm{E}_\alpha^i = e^{t \mathrm{E}_\alpha}$$

si ha

$$\sum_0^\infty {}_i \frac{t^i}{i!} \mathrm{E}_\alpha^i G_m^{(\alpha)}(x;\, u) = e^{t \mathrm{E}_\alpha} G_m^{(\alpha)}(x;\, u) \, .$$

Ma

$$\mathrm{E}_\alpha^i G_m^{(\alpha)}(x;\, u) = G_m^{(\alpha+i)}(x;\, u) \, ,$$

$$e^{t \mathrm{E}_\alpha} G_m^{(\alpha)}(x;\, u) = e^t e^{t \Delta_\alpha} G_m^{(\alpha)}(x;\, u) =$$

$$= e^t \sum_0^m {}_i \frac{t^i}{i!} \Delta_\alpha^i G_m^{(\alpha)}(x;\, u) = e^t \sum_0^m {}_i \frac{(-t)^i}{i!} \mathrm{D}_x^i G_m^{(\alpha)}(x;\, u) \, ,$$

da cui

$$e^{t \mathrm{E}_\alpha} G_m^{(\alpha)}(x;\, u) = e^t G_m^{(\alpha)}(x - t;\, u) \, . \tag{4.23}$$

Quindi vale lo sviluppo

$$\sum_0^\infty {}_i \frac{t^i}{i!} G_m^{(\alpha+i)}(x;\, u) = e^t G_m^{(\alpha)}(x - t;\, u) \, . \tag{4.24}$$

Lo stesso si può stabilire con altro procedimento che riteniamo utile riportare.

Dallo sviluppo elementare

$$\sum_0^\infty {}_i \frac{(tx)^i}{i!} = e^{tx}$$

si ha

$$\sum_0^\infty {}_i \frac{t^i}{i!} x^{-\alpha} e^x U^m \left(\underleftarrow{x^u \mathrm{D}}\right)^m (x^{\alpha+i} e^{-x}) = x^{-\alpha} e^x U^m \left(\underleftarrow{x^u \mathrm{D}}\right)^m \left[x^\alpha e^{-x(1-t)}\right] .$$

Il primo membro equivale a

$$\sum_0^\infty {}_i \frac{(tx)^i}{i!} G_m^{(\alpha+i)}(x;\, u) \, ,$$

il secondo si può trasformare in

$$e^{tx}[x(1-t)]^{-\alpha}\, e^{x(1-t)}\,\cdot$$

$$\cdot[x(1-t)]^{m(1-u)}\underleftarrow{\left[x^u(1-t)^u\,\mathrm{D}_{x(1-t)}\right]^m}\left[x^\alpha(1-t)^\alpha\, e^{-x(1-t)}\right]$$

ed equivale a

$$e^{tx}\, G_m^{(\alpha)}(x-tx;\ u)\ .$$

Pertanto risulta risstabilito lo sviluppo (4.24).

In esso sostituiamo t con $t\,\mathrm{E}_\beta$ e lo applichiamo al polinomio $G_n^{(\beta)}(y;v)$. Si ha

$$\sum_0^\infty{}_i \frac{t^i\,\mathrm{E}_\beta^i}{i!}\, G_m^{(\alpha+i)}(x;\ u)\, G_n^{(\beta)}(y;\ v) = e^{t\,\mathrm{E}_\beta}\, G_m^{(\alpha)}(x-t\,\mathrm{E}_\beta;\ u)\, G_n^{(\beta)}(y;\ v)\ .$$

Il primo membro equivale a

$$\sum_0^\infty{}_i \frac{t^i}{i!}\, G_m^{(\alpha+i)}(x;\ u)\, G_n^{(\beta+i)}(y;\ v)\ .$$

Il secondo a

$$G_m^{(\alpha)}(x-t-t\,\Delta_\beta;\ u)\, e^{t\,\mathrm{E}_\beta}\, G_n^{(\beta)}(y;\ v)\ ,$$

$$e^t\, G_m^{(\alpha)}(x-t-t\,\Delta_\beta;\ u)\ \ G_n^{(\beta)}(y-t;\ v)\ ,$$

$$e^t \sum_0^m{}_i \frac{(-t\,\Delta_\beta)^i}{i!}\, \mathrm{D}_{x-t}^i\, G_m^{(\alpha)}(x-t;\ u)\, G_n^{(\beta)}(y-t;\ v)\ ,$$

$$e^t \sum_0^m{}_i \frac{(t\,\Delta_\beta)^i}{i!}\, \Delta_\alpha^i\, G_m^{(\alpha)}(x-t;\ u)\, G_n^{(\beta)}(y-t;\ v)\ ,$$

$$e^t \sum_0^{\min(m,n)}{}_i \frac{t^i}{i!}\, \Delta_\alpha^i\, G_m^{(\alpha)}(x-t;\ u)\, \Delta_\beta^i\, G_n^{(\beta)}(y-t;\ v)\ .$$

E, a conclusione, vale lo sviluppo

$$(4.25)\qquad \left\{\begin{aligned} &\sum_0^\infty{}_i \frac{t^i}{i!}\, G_m^{(\alpha+i)}(x;\ u)\, G_n^{(\beta+i)}(y;\ v) = \\ &\qquad = e^t \sum_0^{\min(m,n)}{}_i \frac{t^i}{i!}\, \Delta_\alpha^i\, G_m^{(\alpha)}(x-t;\ u)\, \Delta_\beta^i\, G_n^{(\beta)}(y-t;\ v)\ .\end{aligned}\right.$$

Il gruppo di formule operatorie (3.9), ..., (3.12), (3.14), ..., (3.17), (3.19), (3.20), (3.23), (3.24), per la (1.4) e le analoghe riportate all'inizio del n. 4, si traduce facilmente nel seguente gruppo di formule per i polinomi (G):

$$\left\{\begin{aligned} G_n^{(\alpha)}(x;\,u) &= \sum_1^n{}_m\,(1-u)^{n-m}\,a_{n,m}^{(\lambda)}\,G_m^{(\alpha)}(x;\,v) = \\ &= \sum_1^n{}_m\,(1-v)^{n-m}\,\alpha_{n,m}^{(\mu)}\,G_m^{(\lambda)}(x;\,v)\,, \end{aligned}\right. \tag{4.26}$$

$$\left\{\begin{aligned} G_n^{(\lambda)}(x;\,u) &= \sum_0^n{}_m\,(1-u)^{n-m}\,c_{n+1,\,m+1}^{(\lambda)}\,G_m^{[\alpha+m(1-v)]}(x;\,v) = \\ &= \sum_0^n{}_m\,(v-1)^{n-m}\,\beta_{n+1,\,m+1}^{(\mu)}\,G_m^{[\lambda+m(1-v)]}(x;\,v)\,, \end{aligned}\right. \tag{4.27}$$

$$\left\{\begin{aligned} &G_n^{(\alpha)}(x;\,u) = \\ &= \sum_0^n{}_m\,\frac{1}{(1-v)^m}\,\frac{1}{m!}\left[\Delta_x^m\left((1-v)\,x,\,u-1,\,n\right),\ x=\frac{v}{v-1}\right]G_m^{(\alpha+v)}(x;\,v)\,, \end{aligned}\right. \tag{4.28}$$

$$\left\{\begin{aligned} &G_n^{(\lambda)}(x;\,u) = \\ &= \sum_0^n{}_m\,\frac{1}{(v-1)^m}\,\frac{1}{m!}\left[\Delta_x^m\left((v-1)\,x,\,u-1,\,n\right),\ x=\frac{1}{1-v}\right]G_m^{[\alpha+v+m(1-v)]}(x;\,v), \end{aligned}\right. \tag{4.29}$$

$$\left\{\begin{aligned} G_n^{[\alpha+n(1-u)]}(x;\,u) &= \sum_0^n{}_m\,(u-1)^{n-m}\,b_{n+1,\,m+1}^{(\lambda)}\,G_m^{(\alpha)}(x;\,v) = \\ &= \sum_0^n{}_m\,(1-v)^{n-m}\,\gamma_{n+1,\,m+1}^{(\mu)}\,G_m^{(\lambda)}(x;\,v)\,, \end{aligned}\right. \tag{4.30}$$

$$\left\{\begin{aligned} &G_n^{[\alpha+n(1-u)]}(x;\,u) = \\ &= \sum_0^n{}_m\,\frac{1}{(v-1)^m}\,\frac{1}{m!}\left[\Delta_x^m\left((v-1)\,x,\,1-u,\,n\right),\ x=\frac{u}{1-v}\right]G_m^{[\lambda+v+m(1-v)]}(x;\,v), \end{aligned}\right. \tag{4.31}$$

$$\left\{\begin{aligned} &G_n^{[\alpha+n(1-u)]}(x;\,u) = \\ &= \sum_0^n{}_m\,\frac{1}{(1-v)^m}\,\frac{1}{m!}\left[\Delta_x^m\left((1-v)\,x,\,1-u,\,n\right),\ x=1-\frac{u}{1-v}\right]G_m^{(\alpha+v)}(x;\,v)\,, \end{aligned}\right. \tag{4.32}$$

$$\left\{\begin{aligned} G_n^{(\alpha+u)}(x;\,u) &= \sum_0^n{}_m\,(1-u)^{n-m}\,a_{n+1,\,m+1}^{(\lambda)}\,G_m^{(\alpha+v)}(x;\,v) = \\ &= \sum_0^n{}_m\,(1-v)^{n-m}\,\alpha_{n+1,\,m+1}^{(\mu)}\,G_m^{(\lambda+v)}(x;\,v)\,, \end{aligned}\right. \tag{4.33}$$

$$\left\{\begin{aligned} &G_n^{(\alpha+u)}(x;\,u) = \\ &= \sum_0^n{}_m\,\frac{1}{(1-v)^m}\,\frac{1}{m!}\left[\Delta_x^m\left((1-v)\,x,\,u-1,\,n\right),\ x=\frac{u}{1-v}\right]G_m^{(\lambda)}(x;\,v)\,, \end{aligned}\right. \tag{4.34}$$

$$(4.35)\quad \left\{\begin{array}{l} G_n^{(\alpha+u)}(x;\, u) = \\ = \sum_0^n{}_m \dfrac{1}{(v-1)^m}\dfrac{1}{m!}\left[\Delta_x^m\big((v-1)x,\, u-1,\, n\big),\ x = 1-\dfrac{u}{1-v}\right] G_m^{[\alpha+m(1-v)]}(x;\, v), \end{array}\right.$$

$$(4.36)\quad \left\{\begin{array}{l} G_n^{[\alpha+u+n(1-u)]}(x;\, u) = \\ = \sum_0^n{}_m \dfrac{1}{(v-1)^m}\dfrac{1}{m!}\left[\Delta_x^m\big((v-1)x,\, 1-u,\, n\big),\ x = \dfrac{v}{v-1}\right] G_m^{[\alpha+m(1-v)]}(x;\, v)\,, \end{array}\right.$$

$$(4.37)\quad \left\{\begin{array}{l} G_n^{[\alpha+u+n(1-u)]}(x;\, u) = \\ = \sum_0^n{}_m \dfrac{1}{(1-v)^m}\dfrac{1}{m!}\left[\Delta_x^m\big((1-v)x,\, 1-u,\, n\big),\ x = \dfrac{1}{1-v}\right] G_m^{(\alpha)}(x;\, v)\,. \end{array}\right.$$

Le (3.13), (3.18), (3.21), (3.22) conducono a formule che per la (4.18) si possono fare rientrare nel precedente gruppo.

Dalle formule operatorie (3.25), (3.26), (3.27); (3.28), (3.29); (3.30), (3.31) si deducono le formule:

$$(4.38)\qquad G_n^{(\alpha+\beta)}(x;\, u) = \sum_0^n{}_m A_{n+1,\, n-m+1}^{(\beta;\, u)} L_n^{(\alpha-m-1)}(x)\,,$$

$$(4.39)\qquad n!\, G_n^{(\alpha)}(x;\, u) = \sum_1^n{}_m A_{n,m}^{(\mu)}\, G_n^{[\alpha+(m-1)(1-v)]}(x;\, v)\,,$$

$$(4.40)\qquad n!\, G_n^{[\alpha+n(1-u)]}(x;\, u) = \sum_0^n{}_m B_{n+1,\, m+1}^{(\mu)}\, G_n^{[\alpha+m(1-v)]}(x;\, v)\,.$$

E dalle (3.32), (3.33), (3.34) si ricavano le altre formule:

$$(4.41)\qquad G_{2n-1}^{(\alpha)}(x) = \sum_1^n{}_m (1-u)^{2n-2m}\, \underset{2}{k_{n,m}}\, G_{2m-1}^{[\alpha+(m-1)(1-u)]}(x;\, u)\,,$$

$$(4.42)\qquad G_{2n}^{(\alpha)}(x) = \sum_0^n{}_m \left(\frac{1-u}{2}\right)^{2n-2m} T_{n+1,\, m+1}\, G_{2m}^{[\alpha+(m-\frac{1}{2})(1-u)]}(x;\, u)\,,$$

$$(4.43)\qquad 2^n\ n!\, G_n^{(\alpha)}(x) = \sum_0^n{}_m H_{n+1,\, m+1}\, G_n^{[\alpha+(m-\frac{1}{2})(1-u)]}(x;\, u)\,.$$

Tutte le formule precedenti si potrebbero specializzare per i valori di u e v già segnalati che dànno luogo a polinomi particolari o a coefficienti di sviluppo abbastanza semplici. Si ritroverebbero formule note sui polinomi di LAGUERRE, di HERMITE, *attuariali*, e altre sui legami tra i polinomi (G) — generali e particolari — con quelli di LAGUERRE.

Dalla formula di fattorizzazione

$$x^{-\alpha}\, e^{x}\, \underleftarrow{U^{n}\, (\underleftarrow{x^{u}\mathrm{D}})^{n}}\, x^{\alpha}\, e^{-x} = \prod_{0}^{n-1}{}_{m}\, [x\mathrm{D} - x + \alpha + m\,(u-1)]$$

viene subito

$$G_n^{(\alpha)}(x;\, u) = \prod_{0}^{n-1}{}_{m}\, [\underleftarrow{x\mathrm{D} - x + \alpha + m\,(u-1)}] \cdot 1\,. \tag{4.44}$$

E da questa si ha pure

$$G_{n+1}^{(\alpha)}(x;\, u) = [x\mathrm{D} - x + \alpha + n\,(u-1)]\, G_n^{(\alpha)}(x;\, u)\,. \tag{4.45}$$

Dalla formula di potenza

$$\underleftarrow{x^{-\alpha}\, e^{x}\, (x^{u}\mathrm{D})^{n}\, x^{\alpha}\, e^{-x}} = (\underleftarrow{x^{u}\mathrm{D} - x^{u} + \alpha\, x^{u-1}})^{n}$$

si ha:

$$G_n^{(\alpha)}(x;\, u) = U^{n}\, (\underleftarrow{x^{u}\mathrm{D} - x^{u} + \alpha\, x^{u-1}})^{n} \cdot 1 \tag{4.46}$$

e ancora

$$\left\{\begin{aligned} & x^{(m+n)(u-1)}\, G_{m+n}^{(\alpha)}(x;\, u) = \\ & = (\underleftarrow{x^{u}\mathrm{D} - x^{u} + \alpha\, x^{u-1}})^{m}\, [x^{n\,(u-1)}\, G_n^{(\alpha)}(x;\, u)]\,. \end{aligned}\right. \tag{4.47}$$

Inoltre è

$$x^{-\alpha}\, e^{x}\, \underleftarrow{U^{n}\, (\underleftarrow{x^{u}\mathrm{D}})^{n}}\, x^{\alpha}\, e^{-x} = \sum_{0}^{n}{}_{m} \binom{n}{m}\, G_m^{(\alpha)}(x;\, u)\, U^{n-m}\, (\underleftarrow{x^{u}\mathrm{D}})^{n-m}\,. \tag{4.48}$$

I polinomi $G_n^{(\alpha)}(x;\, u)$ sono di grado n tanto in x che in α e dalla formula (4.20) si ha

$$G_n^{(\alpha)}(x;\, u) = \sum_{0}^{n}{}_{m} \binom{n}{m}\, G_{n-m}^{(0)}(x;\, u)\, (\alpha,\, u-1,\, m)\,.$$

Qui appresso li considereremo rispetto al parametro α e stabiliremo per essi le nuove rappresentazioni:

$$(4.49) \qquad G_n^{(\alpha)}(x;\, u) = (-1)^n \Gamma(\alpha+1)\, x^{n(1-u)-\alpha} [\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha - u + 1,\, u)]^n \frac{x^\alpha}{\Gamma(\alpha+1)},$$

$$(4.50) \qquad \begin{cases} G_n^{(\alpha)}(x;\, u) = (-1)^n \Gamma(\alpha+1)\, x^{-\alpha} \cdot \\ \cdot\, \mathrm{E}_\alpha^{n(1-u)} [\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha + (n-1)(u-1),\, u)]^n \dfrac{x^\alpha}{\Gamma[\alpha + n(u-1) + 1]}. \end{cases}$$

Prendiamo le mosse da due operatori permutabili di secondo ordine A e X, di cui A fondamentale, per cui lo scarto $\mathrm{AX} - \mathrm{XA}$ equivale all'operazione identica. E richiamiamo i due sviluppi [26]:

$$U^n (\mathrm{X}^u \mathrm{A})^n = \sum_0^n {}_m (-1)^{n-m} \beta_{n+1,\, m+1}^{(u)} \mathrm{A}^m \mathrm{X}^m,$$

$$\underleftarrow{(\mathrm{A}\,\mathrm{X}^u)^n\, U^n} = \sum_0^n {}_m \beta_{n+1,\, m+1}^{(u)} \mathrm{X}^m \mathrm{A}^m, \qquad \text{con } U = X^{1-u}.$$

Nel primo identifichiamo A con la derivata D_x, X con la moltiplicazione per x, e l'applichiamo alla funzione $x^\alpha e^{-x}$. Si ha

$$(-1)^n G_n^{(\alpha)}(x;\, u) = \sum_0^n {}_m (-1)^m m!\; \beta_{n+1,\, m+1}^{(u)} L_m^{(\alpha)}(x)$$

che, per la

$$L_m^{(\alpha)}(x) = \frac{(-1)^m \Gamma(\alpha + m + 1)}{m!\; x^\alpha} \Delta_\alpha^m \frac{x^\alpha}{\Gamma(\alpha+1)},$$

ci dà

$$(-1)^n x^\alpha G_n^{(\alpha)}(x;\, u) = \Gamma(\alpha+1) \Big[\sum_0^n {}_m \beta_{n+1,\, m+1}^{(u)} (\alpha+1,\, m)\, \Delta_\alpha^m\Big] \frac{x^\alpha}{\Gamma(\alpha+1)}.$$

Nel secondo identifichiamo A con ∇_α definito dalla $\nabla_\alpha f(\alpha) = f(\alpha) - f(\alpha - 1)$, X con $\underleftarrow{\mathrm{E}_\alpha \alpha}$, e osserviamo che

$$(\underleftarrow{\mathrm{E}_\alpha\, \alpha})^m\, \nabla_\alpha^m = (\alpha+1,\, m)\, \mathrm{E}_\alpha^m\, \nabla_\alpha^m = (\alpha+1,\, m)\, \Delta_\alpha^m.$$

Si ottiene

$$[\nabla_\alpha (\underleftarrow{\mathrm{E}_\alpha\, \alpha})^u]^n\, (\underleftarrow{\mathrm{E}_\alpha \alpha})^{n(1-u)} = \sum_0^n {}_m \beta_{n+1,\, m+1}^{(u)} (\alpha+1,\, m)\, \Delta_\alpha^m.$$

Allora per confronto viene

$$G_n^{(\alpha)}(x;\, u) = (-1)^n \Gamma(\alpha+1)\, x^{-\alpha} \left[\nabla_\alpha \underleftarrow{(\mathrm{E}_\alpha \alpha)}^u\right]^n \underleftarrow{(\mathrm{E}_\alpha \alpha)}^{n(1-u)} \frac{x^\alpha}{\Gamma(\alpha+1)} .$$

E poichè

$$\left[\nabla_\alpha \underleftarrow{(\mathrm{E}_\alpha \alpha)}^u\right]^n \underleftarrow{(\mathrm{E}_\alpha \alpha)}^{n(1-u)} \frac{x^\alpha}{\Gamma(\alpha+1)} =$$

$$= \left[\nabla_\alpha \underleftarrow{(\mathrm{E}_\alpha \alpha)}^u\right]^n (\alpha+1,\ n(1-u))\, \mathrm{E}_\alpha^{n(1-u)} \frac{x^\alpha}{\Gamma(\alpha+1)} =$$

$$= x^{n(1-u)} \left[\underleftarrow{\nabla_\alpha \mathrm{E}_\alpha^u (\alpha - u + 1,\ u)}\right]^n \frac{x^\alpha}{\Gamma(\alpha+1)} =$$

$$= x^{n(1-u)} \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha - u + 1,\ u)}\right]^n \frac{x^\alpha}{\Gamma(\alpha+1)} ,$$

resta pertanto stabilita la (4.49).

Osserviamo ora che

$$\mathrm{E}_\alpha^{n(1-u)} \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha + (n-1)(u-1),\ u)}\right]^n \mathrm{E}_\alpha^{n(u-1)} =$$

$$= E_\alpha^{n(1-u)} \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha + (n-1)(u-1),\ u)}\right]^{n-1} \left[\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha + (n-1)(u-1),\ u)\, \mathrm{E}_\alpha^{n(u-1)}\right] =$$

$$= \mathrm{E}_\alpha^{n(1-u)} \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha + (n-1)(u-1),\ u)}\right]^{n-1} \mathrm{E}_\alpha^{n(u-1)} \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha - u + 1,\ u)}\right] =$$

$$= \mathrm{E}_\alpha^{n(1-u)} \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha + (n-1)(u-1),\ u)}\right]^{n-2} \mathrm{E}_\alpha^{n(u-1)} \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha - u + 1,\ u)}\right]^2 =$$

$$= \ldots = \mathrm{E}_\alpha^{n(1-u)} \mathrm{E}_\alpha^{n(u-1)} \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha - u + 1,\ u)}\right]^n = \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha^{u-1} (\alpha - u + 1,\ u)}\right]^n .$$

Per cui risalendo alla (4.49) si deduce la (4.50).

Dalla (4.49) per $u = 1$ si ottiene la formula già da noi stabilita sui polinomi *attuariali* [23]:

$$G_n^{(\alpha)}(x) = \frac{(-1)^n \Gamma(\alpha+1)}{x^\alpha} \underleftarrow{(\Delta_\alpha \alpha)}^n \frac{x^\alpha}{\Gamma(\alpha+1)} .$$

Per ottenere la corrispondente sui polinomi di LAGUERRE facciamo $u = 2$ e sostituiamo α con $\alpha + 1$. Si ha:

$$G_n^{(\alpha+1)}(x;\,2) = n!\, L_n^{(\alpha)}(x) =$$

$$= \frac{(-1)^n \Gamma(\alpha+2)}{x^{\alpha+n+1}} \left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha \alpha(\alpha+1)}\right]^n \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\,.$$

E con la nostra formula [**18**]

$$\left[\underleftarrow{\Delta_\alpha \mathrm{E}_\alpha \alpha(\alpha+1)}\right]^n = (\alpha+2,\, n-1)\, \mathrm{E}_\alpha^n \underleftarrow{\Delta_\alpha^n(\alpha - n + 1,\, n+1)}$$

ritroviamo la

$$L_n^{(\alpha)}(x) = \frac{(-1)^n \Gamma(\alpha+n+1)}{n!\, x^\alpha}\, \Delta_\alpha^n \frac{x^\alpha}{\Gamma(\alpha+1)}\,.$$

5. - I polinomi $G_n^{(\alpha,\,\nu)}(x)$.

La rappresentazione dei polinomi di LAGUERRE rispetto al parametro α, che abbiamo or sopra richiamata, ci ha suggerito nel 1956 l'introduzione e lo studio dei polinomi più generali (1.3).

In parallelo estendiamo qui gli *attuariali* con i più generali $G_n^{(\alpha,\,\nu)}(x)$ definiti dalla (1.15).

A partire dalla formula operatoria [**18**]:

$$\frac{1}{\nu^n}\left(\underleftarrow{\Delta_{\underset{\nu}{\alpha}}\alpha}\right)^n = \sum_0^n{}_m\, k_{n+1,\,m+1}\left(\frac{\alpha}{\nu}+1,\, m\right)\Delta^m_{\underset{\nu}{\alpha}}$$

si ha subito

$$G_n^{(\alpha,\,\nu)}(x) = \sum_0^n{}_m\, (-1)^{n-m} m!\, k_{n+1,\,m+1} \frac{\Gamma\big((\alpha/\nu)+m+1\big)}{\Gamma(\alpha+m\nu+1)} L_m^{(\alpha,\,\nu)}(x)\,. \tag{5.1}$$

D'altra parte è esplicitamente [**24**]:

$$L_n^{(\alpha,\,\nu)}(x) = \sum_0^n{}_i\, \frac{(-1)^i}{n!}\binom{n}{i}\frac{\Gamma(\alpha+n\nu+1)}{\Gamma(\alpha+i\nu+1)}\, x^{i\nu}\,,$$

per cui

$$G_n^{(\alpha,\,\nu)}(x) = (-1)^n\, \Gamma\left(\frac{\alpha}{\nu}+1\right) \sum_0^n{}_i\, \frac{(-x^\nu)^i}{\Gamma(\alpha+i\nu+1)} \sum_i^n{}_m\, (-1)^m \binom{m}{i} k_{n+1,\,m+1}\left(\frac{\alpha}{\nu}+1,\, m\right).$$

Se $\nu = 1$ viene:

$$G_n^{(\alpha)}(x) = (-1)^n \sum_0^n{}_i \frac{(-x)^i}{(\alpha+1,\, i)} \sum_i^n{}_m (-1)^m \binom{m}{i} k_{n+1,\, m+1}(\alpha+1,\, m)\,,$$

mentre è noto che [23]:

$$G_n^{(\alpha)}(x) = \sum_0^n{}_i\, k_{n+1,\, i+1}^{(\alpha)} (-x)^i .$$

Allora risulta per confronto

$$k_{n+1,\, i+1}^{(\alpha)} = \frac{1}{(\alpha+1,\, i)} \sum_i^n{}_m (-1)^{n-m} \binom{m}{i} k_{n+1,\, m+1}(\alpha+1,\, m)\,,$$

e risalendo ai polinomi più generali si ha esplicitamente

$$G_n^{(\alpha,\nu)}(x) = \sum_0^n{}_i\, k_{n+1,\, i+1}^{(\alpha/\nu)} \frac{\Gamma\big((\alpha/\nu)+i+1\big)}{\Gamma(\alpha+i\nu+1)} (-x^\nu)^i . \tag{5.2}$$

Nella somma della serie generatrice dei polinomi $L_n^{(\alpha,\nu)}(x)$,

$$\sum_0^\infty{}_n \frac{t^n}{\Gamma(\alpha+n\nu+1)} L_n^{(\alpha,\nu)}(x)\,,$$

figura la funzione di Bessel-Maitland-Wright

$$\sum_0^\infty{}_n \frac{(-tx^\nu)^n}{n!\ \Gamma(\alpha+n\nu+1)}\,;$$

mentre in quella della serie dei polinomi $G_n^{(\alpha,\nu)}(x)$,

$$\sum_0^\infty{}_n \frac{t^n}{n!} G_n^{(\alpha,\nu)}(x)\,,$$

si presenta la nuova serie

$$\sum_0^\infty{}_n \frac{\Gamma\big((\alpha/\nu)+n+1\big)}{n!\,\Gamma(\alpha+n\nu+1)} [x^\nu(1-e^t)]^i$$

che estende anch'essa quella esponenziale e quella di BESSEL. Si ha:

$$\sum_0^\infty{}_n \frac{t^n}{n!} G_n^{(\alpha,\nu)}(x) =$$

$$= \sum_0^\infty{}_n \frac{t^n}{n!} \sum_0^n{}_i k_{n+1,\,i+1}^{(\alpha/\nu)} \frac{\Gamma\big((\alpha/\nu)+i+1\big)}{\Gamma(\alpha+i\nu+1)} (-x^\nu)^i =$$

$$= \sum_0^\infty{}_i \sum_0^\infty{}_s k_{i+s+1,\,i+1}^{(\alpha/\nu)} \frac{\Gamma\big((\alpha/\nu)+i+1\big)}{(i+s)!\,\Gamma(\alpha+i\nu+1)} t^{i+s}(-x^\nu)^i =$$

$$= \sum_0^\infty{}_i \frac{\Gamma\big((\alpha/\nu)+i+1\big)}{i!\,\Gamma(\alpha+i\nu+1)} (-tx^\nu)^i \sum_0^\infty{}_s \frac{i!}{(i+s)!} k_{i+s+1,\,i+1}^{(\alpha/\nu)} t^s .$$

E poichè la somma rispetto all'indice s è uguale a [21]

$$e^{(\alpha/\nu)t}\left(\frac{e^t-1}{t}\right)^i ,$$

segue a conclusione

$$\sum_0^\infty{}_n \frac{t^n}{n!} G_n^{(\alpha,\nu)}(x) = e^{(\alpha/\nu)t} \sum_0^\infty{}_n \frac{\Gamma\big((\alpha/\nu)+n+1\big)}{n!\,\Gamma(\alpha+n\nu+1)} [x^\nu(1-e^t)]^n . \tag{5.3}$$

Per $\nu=1$ si ritrova la funzione generatrice dei polinomi $G_n^{(\alpha)}(x)$. Per $\nu=2$ si ha

$$\sum_0^\infty{}_n \frac{t^n}{n!} G_n^{(\alpha,2)}(x) =$$

$$= e^{(\alpha/2)t} \sum_0^\infty{}_n \frac{\Gamma\big((\alpha/2)+n+1\big)}{n!\,\Gamma(\alpha+2n+1)} [x^2(1-e^t)]^n =$$

$$= \frac{\Gamma\big((\alpha/2)+1\big)}{\Gamma(\alpha+1)} e^{(\alpha/2)t} \sum_0^\infty{}_n \frac{\big((\alpha/2)+1,\,n\big)}{(\alpha+1,\,2n)} \frac{[x^2(1-e^t)]^n}{n!} =$$

$$= \frac{\Gamma\big((\alpha/2)+1\big)}{\Gamma(\alpha+1)} e^{(\alpha/2)t} \sum_0^\infty{}_n \frac{[(x^2/4)(1-e^t)]^n}{n!\,\big((\alpha+1)/2,\,n\big)} .$$

Con la funzione di BESSEL è

$$\sum_0^\infty{}_n \frac{[(x^2/4)\,(1-e^t)]^n}{n!\,\big((\alpha+1)/2,\,n\big)} = \Gamma\left(\frac{\alpha+1}{2}\right)\left[\frac{x^2}{4}\,(e^t-1)\right]^{(1-\alpha)/4} J_{(\alpha-1)/2}\big(x\sqrt{e^t-1}\big)\,.$$

Inoltre

$$\frac{\Gamma\big((\alpha+1)/2\big)\;\Gamma\big((\alpha/2)+1\big)}{\Gamma(\alpha+1)} = \frac{\sqrt{\pi}}{2^\alpha}\,.$$

Pertanto

$$(5.4)\qquad \left\{\begin{aligned} &\sum_0^\infty{}_n \frac{t^n}{n!}\,G_n^{(\alpha,\,2)}(x) = \\ &= \frac{\sqrt{\pi}}{2^{(\alpha+1)/2}}\,e^{(\alpha/2)t}\,[x^2\,(e^t-1)]^{(1-\alpha)/4}\,J_{(\alpha-1)/2}\big(x\sqrt{e^t-1}\big). \end{aligned}\right.$$

E se facciamo ancora $\alpha = 1$ si ottiene lo sviluppo:

$$(5.5)\qquad \sum_0^\infty{}_n \frac{t^n}{n!}\,G_n^{(1,\,2)}(x) = \frac{\sqrt{\pi}}{2}\,e^{t/2}J_0\big(x\sqrt{e^t-1}\,\big),$$

generatore dei polinomi

$$G_n^{(1,\,2)}(x) = \frac{\sqrt{\pi}}{2}\sum_0^n{}_m \frac{1}{m!}\,k_{n+1,\,m+1}^{(1/2)}\,(-\,x^2/4)^m\,,$$

Dalla (5.3), per derivazione rispetto a x^ν, si ha:

$$\sum_1^\infty{}_n \frac{t^n}{n!}\,\mathrm{D}_{x^\nu}\,G_n^{(\alpha,\,\nu)}(x) =$$

$$= e^{(\alpha/\nu)t}\,(1-e^t)\sum_0^\infty{}_i \frac{\Gamma\big((\alpha+\nu)/\nu+i+1\big)}{i!\,\Gamma(\alpha+\nu+i\nu+1)}\,[x^\nu\,(1-e^t)]^i =$$

$$= (e^{-t}-1)\sum_0^\infty{}_s \frac{t^s}{s!}\;G_s^{(\alpha+\nu,\,\nu)}(x)\;.$$

Quindi:

$$\sum_0^\infty{}_m \frac{t^{m+1}}{(m+1)!}\,\mathrm{D}_{x^\nu}\,G_{m+1}^{(\alpha,\,\nu)}(x) = \sum_0^\infty{}_r \frac{(-\,t)^{r+1}}{(r+1)!}\,\sum_0^\infty{}_s \frac{t^s}{s!}\,G_s^{(\alpha+\nu,\,\nu)}\,(x) =$$

$$= \sum_0^\infty{}_m \frac{t^{m+1}}{(m+1)!}\sum_0^m{}_s\,(-1)^{m-s+1}\binom{m+1}{s}\;G_s^{(\alpha+\nu,\,\nu)}(x)\;.$$

E in conseguenza risulta ($n \geq 1$):

(5.6)
$$\mathrm{D}_{x^\nu}\, G_n^{(\alpha,\,\nu)}(x) = \sum_0^{n-1}{}_s\, (-1)^{n-s} \binom{n}{s} G_s^{(\alpha+\nu,\,\nu)}(x)\,.$$

Per derivazione rispetto a t si ha:

$$\sum_0^\infty{}_n \frac{t^n}{n!} G_{n+1}^{(\alpha,\,\nu)}(x) = \frac{\alpha}{\nu}\sum_0^\infty{}_n \frac{t^n}{n!} G_n^{(\alpha,\,\nu)}(x) -$$

$$- x^\nu e^{(\alpha+\nu)t/\nu} \sum_0^\infty{}_n \frac{\Gamma\big((\alpha+\nu)/\nu + n + 1\big)}{n!\,\Gamma(\alpha+\nu+n\nu+1)} [x^\nu(1-e^t)]^n =$$

$$= \frac{\alpha}{\nu}\sum_0^\infty{}_n \frac{t^n}{n!} G_n^{(\alpha,\,\nu)}(x) - x^\nu \sum_0^\infty{}_n \frac{t^n}{n!} G_n^{(\alpha+\nu,\,\nu)}(x)\,,$$

da cui

(5.7)
$$G_{n+1}^{(\alpha,\,\nu)}(x) = \frac{\alpha}{\nu} G_n^{(\alpha,\,\nu)}(x) - x^\nu G_n^{(\alpha+\nu,\,\nu)}(x)\,.$$

Concludiamo qui con l'inversa della (5.1), che si può dedurre dalla formula operatoria [**18**],

$$\left(\frac{\alpha}{\nu}+1,\, n\right) \mathop{\Delta}_{\nu}{}_{\alpha}^{n} = \sum_0^n{}_m \frac{1}{\nu^m} h_{n+1,\,m+1} \big(\mathop{\Delta_\alpha}_{\nu}\alpha\big)^m\,,$$

ed è

(5.8)
$$n!\,\frac{\Gamma\big((\alpha/\nu)+n+1\big)}{\Gamma(\alpha+n\nu+1)} L_n^{(\alpha,\,\nu)}(x) = \sum_0^n{}_m (-1)^{n-m} h_{n+1,\,m+1} G_m^{(\alpha,\,\nu)}(x)\,.$$

6. – Le espressioni esplicite dei polinomi $G_n^{(\alpha)}(x;\,u)$ e $G_n^{(\alpha,\,\nu)}(x)$ per $n=0, 1, 2, 3$ sono le seguenti:

$$G_0^{(\alpha)}(x;\,u) = 1\,, \qquad G_1^{(\alpha)}(x;\,u) = \alpha - x\,,$$

$$G_2^{(\alpha)}(x;\,u) = \alpha(\alpha+u-1) - (2\alpha+u)x + x^2,$$

$$G_3^{(\alpha)}(x;\,u) = \alpha(\alpha+u-1)(\alpha+2u-2) - (3\alpha^2+6\alpha u+2u^2-3\alpha-u)x +$$
$$+ 3(\alpha+u)x^2 - x^3;$$

$$G_0^{(\alpha,\nu)}(x) = \frac{\Gamma\big((\alpha/\nu)+1\big)}{\Gamma(\alpha+1)}\,,$$

$$G_1^{(\alpha,\nu)}(x) = \frac{\alpha}{\nu}\,\frac{\Gamma\big((\alpha/\nu)+1\big)}{\Gamma(\alpha+1)} - \frac{\Gamma\big((\alpha+\nu)/\nu+1\big)}{\Gamma(\alpha+\nu+1)}\,x^\nu\,,$$

$$G_2^{(\alpha,\nu)}(x) = \frac{\alpha^2}{\nu^2}\,\frac{\Gamma\big((\alpha/\nu)+1\big)}{\Gamma(\alpha+1)} - \frac{2\alpha+\nu}{\nu}\,\frac{\Gamma\big((\alpha+\nu)/\nu+1\big)}{\Gamma(\alpha+\nu+1)}\,x^\nu + \frac{\Gamma\big((\alpha+2\nu)/\nu+1\big)}{\Gamma(\alpha+2\nu+1)}\,x^{2\nu}\,,$$

$$G_3^{(\alpha,\nu)}(x) = \frac{\alpha^3}{\nu^3}\,\frac{\Gamma\big((\alpha/\nu)+1\big)}{\Gamma(\alpha+1)} - \frac{3\alpha^2+3\alpha\nu+\nu^2}{\nu^2}\,\frac{\Gamma\big((\alpha+\nu)/\nu+1\big)}{\Gamma(\alpha+\nu+1)}\,x^\nu +$$

$$+ \frac{3(\alpha+\nu)}{\nu}\,\frac{\Gamma\big((\alpha+2\nu)/\nu+1\big)}{\Gamma(\alpha+2\nu+1)}\,x^{2\nu} - \frac{\Gamma\big((\alpha+3\nu)/\nu+1\big)}{\Gamma(\alpha+3\nu+1)}\,x^{3\nu}\,.$$

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Summary.

Contributions to the theory of certain generalized Laguerre polynomials and actuarial polynomials are established.

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