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On Sequence of Contraction Mappings. (**)

1 . – Let X be a metric space. A mapping T of the space X into itself is said to be a contraction map if there exists a number k such that

$$d(Tx, Ty) \leqslant k\, d(x, y),$$

for any two points $x, y \in X$, where $0 \leqslant k < 1$. Every contraction map is continuous.

The classical contraction mapping principle of Banach states that if (X, d) is a complete metric space and T: $X \to X$ is a contraction mapping, then T has a unique fixed point.

Contraction mappings on metric spaces have been of great interest for many years. In the present paper we study a sequence of contraction mappings and fixed points. An application to differential equation has also been given.

A question to ask is the following:

In a complete metric space does the convergence of a sequence of contraction mappings to a contraction mapping T imply the convergence of the sequence of their fixed points to the fixed point of T ? [3].

A partial answer to this question has been given [1]. « Let X be a complete metric space, and let T and T_n $(n = 1, 2, ...)$ be contraction mappings of X into itself with the same Lipschitz constant $k < 1$, and with fixed points U and U_n respectively. Suppose that $\lim_{n \to \infty} T_n(x) = T(x)$ for every $x \in X$. Then $\lim_{n \to \infty} U_n = U$. » The restriction in this theorem that all the contraction mappings have the « same Lipschitz constant » is very strong.

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2. – We have the following result:

Theorem 1. *Let X be a complete metric space and let T_n $(n=1, 2, \ldots)$ be contraction mapping of X into itself with fixed points U_n and with Lipschitz constants k_n such that $k_{n+1} \leqslant k_n$ for each n.*

Suppose that $\lim_{n \to \infty} T_n x = Tx$ for every $x \in X$, where T is a mapping from X into itself. Then T has a unique fixed point U and $\lim_{n \to \infty} U_n = U$.

Proof. Since $|T_n x - T_n y| \leqslant k_n |x-y|$, therefore

$$\lim_{n \to \infty} |T_n x - T_n y| \leqslant \lim_{n \to \infty} k_n |x-y| .$$

Since $k_{n+1} \leqslant k_n$ for each n, it follows that $\lim k_n < 1$. Hence $\lim_{n \to \infty} T_n x = Tx$ is a contraction mapping. Moreover, k_1 will serve the purpose of a LIPSCHITZ constant for T_n $(n=1, 2, \ldots)$. Thus the proof follows from theorem 1.2 in [1] by replacing k by k_1.

Remark. However, if the LIPSCHITZ constants are such that $k_{n+1} \geqslant k_n$ for each n, the theorem is, in general, false.

In order to illustrate the theorem we take the following example: Consider

$$T_n\colon [0, 2] \to [0, 2], \quad \text{defined by} \quad T_n x = 1 + x/(n+1) \qquad (n=1, 2, \ldots).$$

Then $\lim_{n \to \infty} T_n x = Tx = 1$ for every $x \in [0, 2]$. The LIPSCHITZ constant is $k_n = 1/(n+1)$ $(n=1, 2, \ldots)$. Thus $k_1 = 1/2$ will serve the purpose for all mappings to be contraction. The corresponding fixed point for each T_n is $U_n = (n+1)/n$ $(n=1, 2, \ldots)$. $\operatorname{Lim}_{n \to \infty} U_n = 1$, where $U=1$ is a unique fixed point for T.

3. – As an application of Theorem 1, we give the following proposition due to Professor J. R. DORROH, on the same llnes as given in [3].

Let D be an open subset of the plane, let $(a, b) \in D$, let $M > 0$ be a real number, and let $\{k_i\}$ be a decreasing sequence of positive real numbers. For each $i = 0, 1, 2, \ldots$, let f_i be a real valued continuous function defined on D such that

$$|f_i(x, y)| \leqslant M \qquad \textit{for all} \quad (x, y) \in D,$$

and

$$| f_i(x, y) - f_i(x, z) | \leqslant k_i | y - z | \qquad \text{for all} \qquad (x, y), (x, z) \in D.$$

Suppose also that the sequence $\{f_i\}$ *converges to* f *on* D. *Let* h *be such that* $0 \leqslant k_i h < 1$ *for all* $i = 0, 1, 2, \ldots$, *and such that* $G = \{(x, y)$ *with* $| x - a | < h$ *and* $| y - b | < M | x - a | \}$ *is a subset of* D. *Then the sequence* $\{y_i\}$ *converges on* $I = [a - h, a + h]$ *to* y_0, *where, for each* $i = 0, 1, 2, \ldots$, y_i *is the unique solution on* I *of the initial value problem*

$$\begin{cases} y(a) = b \\ y'(x) = f_i(x, y(x)) . \end{cases}$$

Proof. Let X be the set of all real valued functions defined on I with graph lying in G and with LIPSCHITZ constant less than or equal to M. Then (X, d) is a complete metric space with d as supremum metric. For each $i = 0, 1, 2, \ldots$ and each $g \in X$, define $T_i(g)$ at each $x \in I$ by

$$T_i(g)\, x = b + \int_a^x f_i(t, g(t))\, dt .$$

It can be easily seen that, for each $i = 0, 1, 2, \ldots$, T_i is a contraction mapping from X into itself with LIPSCHITZ constant less than or equal to $k_i h$. For each $g \in X$, $x \in I$ and $i = 1, 2, \ldots$,

$$T_i(g)\, x - T_0(g)\, x = \int_a^x [f_i(t, g(t)) - f_0(t, g(t))]\, dt .$$

Since the sequence of integrands converges pointwise to zero and is uniformly bounded by M, the LEBESGUE bounded convergence theorem guarantees that the sequence of integrals goes to zero is $i \to \infty$. Therefore, the sequence $\{T_i(g)\}$ converges pointwise to $T_0(g)$ on I. This implies by the equicontinuity of $\{T_i(g)\}$ on the compact set I, that the sequence $\{T_i(g)\}$ converges uniformly to $T_0(g)$. Hence, the sequence $\{T_i\}$ converges to T_0 on X. By Theorem 1, the sequence $\{y_i\}$, where y_i is the unique fixed point of T_i for each $i = 1, 2, \ldots$, converges to the fixed point y_0 of T_0. The result follows since these fixed points are the unique solutions of the initial value problem.

Definition 1. A mapping T of X into itself is said to be locally contractive if for every $x \in X$ there exist ϱ and λ ($\varrho > 0$, $0 < \lambda < 1$) which may depend on x such that

$$p,\ q \in s_\varrho(x) = [y/d(x,\ y) < \varrho]$$

implies

$$d(Tp,\ Tq) < \lambda\, d(p,\ q), \qquad p \neq q.$$

Definition 2. Let (X, d) be a metric space and $\varrho > 0$. A finite sequence $x_0, x_1, \ldots, x_n$ of points of X is called ϱ-chain joining x_0 and x_n if

$$d(x_{i-1},\ x_i) < \varrho \qquad (i = 1,\ 2,\ \ldots,\ n).$$

The metric space (X, d) is said to be ϱ-chainable (well-linked) if for each pair (x, y) of its points there exists a ϱ-chain joining x and y.

4. – We prove the following result:

Theorem 2. *Let (X, d) be a complete ϱ-chainable metric space.*
Let T_n: $X \to X$ be a function with at least one fixed point U_n for each $n = 1, 2, \ldots$, and let T: $X \to X$ be a locally contractive mapping with fixed point U. If the sequence $\{T_n\}$ converges uniformly to T, then the sequence $\{U_n\}$ converges to U.

Proof. (X, d) being ϱ-chainable we define, for $x, y \in X$,

$$d_\varrho(x,\ y) = \inf \sum_{i=1}^{n} d(x_{i-1},\ x_i)\,,$$

where the infimum is taken over all ϱ-chains $x_0, x_1, \ldots, x_n$ joining $x_0 = x$ and $x_n = y$. Then d_ϱ is a metric for X satisfying

$$(1) \qquad d(x,\ y) \leqslant d_\varrho(x,\ y)$$

and

$$(2) \qquad d(x,\ y) = d_\varrho(x,\ y) \qquad \text{for} \quad d(x,\ y) < \varrho.$$

From (1), (2) and completeness of (X, d) it follows that (X, d_ϱ) is complete. It can be easily seen that T is a contraction mapping in the metric space (X, d_ϱ) [2].

Let $\varepsilon > 0$ and choose a natural number N such that $i \geqslant N$ implies $d_\varrho(T_i(x), T(x)) < \varepsilon(1-k)$ for all $x \in X$, where $k < 1$ is a LIPSCHITZ constant for T. Then, if $i \geqslant N$,

$$\begin{aligned} d_\varrho(u_i, u) &= d_\varrho(T_i(u_i), T(u)) \\ &\leqslant d_\varrho(T_i(u_i), T(u_i)) + d_\varrho(T(u_i), T(u)) \\ &< \varepsilon(1-k) + k\, d(u_i, u). \end{aligned}$$

Hence, $d_\varrho(u_i, u) < \varepsilon$ for all $i \geqslant N$. This proves that $\{u_i\}$ converges to u.

References.

[1] F. F. BONSALL, *Lectures on Some Fixed Point Theorems of Functional Analysis*, Tata Institute of Fundamental Research, Bombay, India 1962.

[2] M. EDELSTEIN, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. **12** (1961), 7-10.

[3] S. B. NADLER (jr.), *Sequences of contractions and fixed points*, Pacific J. Math. **27** (1968), 579-585.

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