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On Generalised Stirling Numbers and Polynomials. (**)

1. - Introduction.

STIRLING numbers and polynomials are defined as [2]:

$$S(n,\ k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n = \frac{(-1)^k}{k!} \Delta^k\, 0^n \tag{1.1}$$

and

$$A_n(x) = \sum_{k=0}^{n} S(n,\ k)\, x^k . \tag{1.2}$$

R. P. SINGH [1] has given generalisations of (1.1) and (1.2) as:

$$S^{(\alpha)}(n,\ k,\ r) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\alpha + rj)^n , \tag{1.3}$$

$$T_n^{(\alpha)}(x,\ r,\ -p) = \sum_{k=0}^{n} S^{(\alpha)}(n,\ k,\ r)\, p^r x^{rk} . \tag{1.4}$$

In particular it is found that

$$S^{(0)}(n,\ k,\ 1) = S(n,\ k) , \qquad T_n^{(0)}(x,\ 1,\ -1) = A_n(x) .$$

A recurrence relation for (1.3) is [1]

$$S^{(\alpha)}(n+1,\ k,\ r) = r\, S^{(\alpha)}(n,\ k-1,\ r) + (\alpha + rk)\, S^{(\alpha)}(n,\ k,\ r) . \tag{1.5}$$

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In the present Note it is proposed to study more properties of (1.3) and (1.4).

2. – It is familiar that the formulas

$$g(n) = \sum_{d/n} f(d) \qquad (n = 1, 2, \ldots) \tag{2.1}$$

and

$$f(n) = \sum_{cd=n} M(c)\, g(d) , \tag{2.2}$$

where $M(c)$ is the MOBIÜS function, are equivalent.

From (2.1) and (2.2) we have that, if

$$g(r) = \sum_{j=0}^{r} \binom{r}{j} f(j) \qquad (r = 0, 1, \ldots) , \tag{2.3}$$

then

$$f(r) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} g(j) . \tag{2.4}$$

From (2.3), (2.4) and (1.3), we deduce that

$$(\alpha + rj)^n = \sum_{i=0}^{j} \binom{j}{i} i!\ S^{(\alpha)}(n, i, r) . \tag{2.5}$$

Further we obtain by simple manipulation that

$$S^{(\alpha+1)}(n, k, r) = \sum_{i=0}^{n} \binom{n}{i} S^{(\alpha)}(i, k, r) \tag{2.6}$$

and

$$S^{(\alpha)}(n+1, k, r) - \alpha\, S^{(\alpha)}(n, k, r) = r \sum_{i=0}^{n} \binom{n}{i} r^{n-i} S^{(\alpha)}(i, k-1, r) . \tag{2.7}$$

A particular case of (2.7) is given as a familiar result

$$S(n+1, k) = \sum_{i=0}^{n} \binom{n}{i} S(i, k-1) . \tag{2.8}$$

For (1.3), generating function is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} S^{(\alpha)}(n, k, r) = \frac{(-1)^k}{k!} e^{\alpha t} (1 - e^{rt})^k \tag{2.9}$$

and (2.9) leads us to addition theorem

$$S^{(\alpha+\beta)}(n, 2k, r) = \frac{(k!)^2}{(2k)!} \sum_{m=0}^{n} \binom{n}{m} S^{(\alpha)}(n-m, k, r)\, S^{(\beta)}(m, k, r)\,. \tag{2.10}$$

3. – Let

$$\Delta_r^k f(\alpha) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(\alpha + rj)\,, \tag{3.1}$$

hence

$$\begin{cases} \Delta_r^i (\alpha + rj)^n = \sum_{s=0}^{i} (-1)^{i-s} \binom{i}{s} (\alpha + rj + rs)^n \\ \qquad = i! \sum_{s=0}^{i} \binom{i}{s} (rs)^{i-s} S^{(\alpha)}(s, i, r)\,. \end{cases} \tag{3.2}$$

From (1.5) we obtain the following congruences (mod 2)

$$S^{(\alpha)}(n+1, 2s, r) \equiv r\, S^{(\alpha)}(n, 2s-1, r) + \alpha\, S^{(\alpha)}(n, 2s, r)\,, \tag{3.3}$$

$$S^{(\alpha)}(n+1, 2s+1, r) \equiv r\, S^{(\alpha)}(n, 2s, r) + (\alpha + r)\, S^{(\alpha)}(n, 2s+1, r), \tag{3.4}$$

$$S^{(\alpha)}(n+1, 2s+1, r) \equiv r^2\, S^{(\alpha)}(n-1, 2s-1, r) + \tag{3.5}$$

$$+ r\alpha\, S^{(\alpha)}(n-1, 2s, r) + (\alpha + r)\, S^{(\alpha)}(n, 2s-1, r)\,,$$

which reduces in particular to [3]

$$C_{n+1,2s} \equiv C_{n,2s-1} + C_{n,2s} \pmod 2,$$

$$C_{n+1,2s+1} \equiv C_{n,2s} \pmod 2,$$

$$C_{n+1,2s} \equiv C_{n-1,2s-2} + C_{n,2s} \pmod 2,$$

where $C_{n,r} = S(n+1, r+1)$.

4. – For STIRLING polynomials we know well that

$$A_n(t)\, e^t = \sum_{j=0}^{\infty} \frac{j^n\, t^j}{j!}\,. \tag{4.1}$$

Hence we obtain

$$\sum_{j=0}^{\infty} \frac{(\alpha + rj)^n}{j!}\, t^j = e^t\,[\alpha + r\, A(t)]^n\,, \tag{4.2}$$

where A^n is to replaced by A_n. Also

$$\sum_{j=0}^{\infty} \frac{(\alpha + rj)^n}{j!}\, t^j = e^t\, T_n^{(\alpha)}(t^{1/r},\ r,\ -1)\,, \tag{4.3}$$

where $T_n^r(x,\ r,\ p)$ are defined by (1.4). Thus from (4.2) and (4.3), we obtain

$$T_n^{(\alpha)}(t^{1/r},\ r,\ -1) = [\alpha + r\, A(t)]^n\,. \tag{4.4}$$

Further it is seen that

$$\sum_{k=0}^{\infty} t^k\, S^{(\alpha)}(n,\ k,\ r) = [\alpha + r\, A(t)]^n \tag{4.5}$$

and

$$T_n^{(\alpha)}(x,\ r,\ -p) = e^{-p x^r} \sum_{j=0}^{\infty} \frac{p^j\, x^{rj}}{j!}\, (\alpha + rj)^n\,, \tag{4.6}$$

which can be verified by manipulating right hand side and this gives an extension to (4.1).

Differentiating $w \cdot r \cdot t \cdot x$ the generating function for (1.3); given by [1]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, T_n^{(\alpha)}(x,\ r,\ -p) = \exp[\alpha t - p\, x^r\,(1 - e^{rt})]\,, \tag{4.7}$$

we obtain

$$\begin{cases} D\, T_n^{(\alpha)}(x,\ r,\ -p) + r\, p\, x^{r-1}\, T_n^{(\alpha)}(x,\ r,\ -p) = \\ \qquad = r\, p\, x^{r-1} \displaystyle\sum_{m=0}^{n} \binom{n}{m}\, r^{n-m}\, T_m^{(\alpha)}(x,\ r,\ -p)\,, \end{cases} \tag{4.8}$$

and on differentiating $w \cdot r \cdot t \cdot t$

$$(4.9)\quad \begin{cases} T_{n+1}^{(\alpha)}(x,\ r,\ -p) = \\ = \alpha\, T_n^{(\alpha)}(x,\ r,\ -p) + r\, p\, x^r \sum_{m=0}^{n} \binom{n}{m} r^{n-m}\, T_m^{(\alpha)}(x,\ r,\ -p)\,. \end{cases}$$

On combining (4.8) and (4.9), we get

$$(4.10)\qquad T_{n+1}^{(\alpha)}(x,\ r,\ -p) = x\,\mathrm{D}\, T_n^{(\alpha)}(x,\ r,\ -p) + (\alpha + r\, p\, x^r)\, T_n^{(\alpha)}(x,\ r, -p)\,.$$

(1.3) can also be written as

$$(4.11)\qquad T_n^{(\alpha)}(x,\ r,\ -p) = \sum_{i=0}^{n} \binom{n}{i} \alpha^{n-i}\, r^i\, A_n(p\, x^r)\,.$$

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References.

[1] R. P. Singh, *On generalised Truesdell polynomials*, Riv. Mat. Univ. Parma (2) **8** (1967), 345-353.

[2] J. Riordan, *An Introduction to the Combinatorial Analysis*, Wiley, New York 1958.

[3] L. Carlitz, *Single variable Bell polynomials*, Collect. Math. Barcelona **14** (1962), 13-25.

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