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On $|\overline{N}, p_n|$ - Summability Factors of Infinite Series. (**)

1.1. – Let $\sum_{n=1}^{\infty} a_n$ be a given infinite series, with the sequence of partial sums $\{s_n\}$, and let $\{p_n\}$ be a sequence of real positive constants such that

$$P_n = \sum_{v=0}^{n} p_v .$$

We write

$$\bar{\sigma}_n^p = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v , \qquad \bar{t}_n^p = \frac{1}{P_n} \sum_{v=1}^{n} P_{v-1} a_v .$$

The series $\sum a_n$ is said to be absolutely summable $(\overline{N}, p_n)$, or summable $|\overline{N}, p_n|$, if $\bar{\sigma}_n^p \in$ B.V.. If we take $p_n = 1$ for all n, then it is known that $|\overline{N}, 1| \sim |C, 1|$, and if we take $p_n = 1/n$, then it is known [3] (1) that $|\overline{N}, 1/n| \sim |R, \log n, 1|$. We write throughout, for any sequence $\{u_n\}$,

$$\Delta u_n = u_n - u_{n+1} , \qquad \Delta^2 u_n = \Delta(\Delta u_n) .$$

1.2. – The object of this paper is to discuss a problem on $|\overline{N}, p_n|$-summability factors which seems to have not been tackled so far. We consider the problem of determining suitable type of sequences $\{\varepsilon_n\}$ such that $\sum \varepsilon_n a_n$ may be summable $|\overline{N}, p_n|$, whenever $\sum a_n$ is not summable $|\overline{N}, p_n|$, but the total variation of $(\overline{N}, p_n)$-mean of $\sum a_n$ is of certain order, say μ_n, where μ_n is positive and non-decreasing.

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(1) This can be easily proved by employing the method used in IYER [3].

We observe that whenever $p_n = 1$ for all n, our theorem includes the following theorem of PATI [5].

Theorem A. Let $\{\lambda_n\}$ be a convex sequence such taht $\sum n^{-1} \lambda_n$ is convergent. If $\sum a_n$ is bounded $[R, \log n, 1]$, then $\sum \lambda_n a_n$ is summable $|C, 1|$.

When $p_n = 1/n$, it is to be noted that the hypothesis of our theorem is weaker than a modified version of the following theorem of KULSHRESHTHA [4].

Theorem B. If $\sum a_n$ is bounded $[R, \log n, 1]$ and if the sequence $\{\lambda_n\}$ satisfies the following conditions:

(a) λ_n is positive, bounded and monotonic non-increasing,

(b) $\sum_{n=2}^{m} \lambda_n/(n \log n) = O(1)$,

(c) $\sum_{n=2}^{m} \log n \cdot \Delta \lambda_{n+1} = O(1)$,

(d) $\sum_{n=2}^{m} n \log n \cdot |\Delta^2 \lambda_n| = O(1)$,

as $m \to \infty$, then $\sum a_n \lambda_n$ is summable $|R, \log n, 1|$.

However, it is natural to get factors heavier than that of Theorem B.

2.1. – We establish the following theorem.

Theorem. *Let* $p_n > 0$ $(n = 0, 1, 2, \ldots)$ *and* $(n+1) p_n \leqslant K P_n$. *If*

$$\sum_{v=1}^{n} (p_v/P_{v-1}) \, |\bar{t}_v^v| = O(\mu_n),$$

where $\{\mu_n\}$ *is a positive monotonic non-decreasing sequence, and if the sequences* $\{\varepsilon_n\}$ *and* $\{\mu_n\}$ *are such that*

(i) (a) $\qquad \varepsilon_n \mu_n = O(1)$,

(i) (b) $\qquad \Delta \mu_n = O\left(\frac{|\Delta p_n|}{p_n} \mu_n\right) \qquad$ *as* $n \to \infty$,

(ii) $\qquad \sum_{n=1}^{\infty} P_n \, |\Delta(1/p_n)| \, \mu_n \, |\Delta \varepsilon_{n+1}| < \infty$,

(iii) $\qquad \sum_{n=1}^{\infty} (P_n/p_n) \, \mu_n \, |\Delta^2 \varepsilon_n| < \infty$,

then the series $\sum \varepsilon_n a_n$ *is summable* $|\bar{N}, p_n|$.

2.2. – We require the following lemmas for the proof of our theorem.

Lemma 1. *Let* $p_n > 0$, *for all* $n \geqslant 0$, *such that* $(n+1)\, p_n \leqslant K P_n$. *If* $\varepsilon_n \mu_n = O(1)$ *and*

$$\sum_{n=1}^{\infty} (P_n/p_n)\, \mu_n \, |\Delta^2 \varepsilon_n| < \infty,$$

where $\{\mu_n\}$ *is a positive monotonic non-decreasing sequence, then*

$$\sum_{n=1}^{\infty} \mu_n \, |\Delta \varepsilon_n| < \infty.$$

Proof. By hypothesis and by a lemma of ANDERSEN [1] (see also [2], Lemma 8), we have

$$\Delta \varepsilon_n = \sum_{v=n}^{\infty} \Delta^2 \varepsilon_v .$$

Now

$$\sum_{n=1}^{\infty} \mu_n \, |\Delta \varepsilon_n| = \sum_{n=1}^{\infty} \mu_n \, \Big|\sum_{v=n}^{\infty} \Delta^2 \varepsilon_v\Big| \leqslant$$

$$\leqslant \sum_{v=1}^{\infty} |\Delta^2 \varepsilon_v| \sum_{n=1}^{v} \mu_n \leqslant \sum_{v=1}^{\infty} |\Delta^2 \varepsilon_v|\, \mu_v (1+v) \leqslant$$

$$\leqslant K \sum_{v=1}^{\infty} (P_v/p_v)\, \mu_v \, |\Delta^2 \varepsilon_v| \leqslant K \ (^2),$$

by hypothesis.

Lemma 2. *If the sequences* $\{\mu_n\}$ *and* $\{\varepsilon_n\}$ *satisfy the same conditions as in the Theorem, then*

$$(P_{n-1}/p_n)\, \mu_n \, |\Delta \varepsilon_n| = O(1), \qquad \text{as } n \to \infty .$$

Proof. Since

$$\sum_{n=1}^{\infty} \left| \Delta\left(\frac{P_{n-1}}{p_n} \mu_n \, |\Delta \varepsilon_n| \right) \right| \leqslant$$

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(2) K denotes throughout an absolute constant, not necessarily the same at each occurrence.

$$\leqslant \sum_{n=1}^{\infty} (P_{n-1}/p_n)\,\mu_n\;|\Delta^2\varepsilon_n| + \sum_{n=1}^{\infty} \mu_n\;|\Delta\varepsilon_{n+1}| + \sum_{n=1}^{\infty} \mu_n P_n\;|\Delta(1/p_n)||\Delta\varepsilon_{n+1}| +$$

$$+ \sum_{n=1}^{\infty} (P_n/p_{n+1})\;|\Delta\mu_n||\Delta\varepsilon_{n+1}|$$

$$\leqslant K + \sum_{n=1}^{\infty} (P_n/p_{n+1})\;|\Delta\mu_n||\Delta\varepsilon_{n+1}| \quad \text{[by hypothesis (ii), (iii) and Lemma 1]}$$

$$\leqslant K + \sum_{n=1}^{\infty} P_n\mu_n\;|\Delta(1/p_n)||\Delta\varepsilon_{n+1}| \quad \text{[by hypothesis (i)]}$$

$$\leqslant K\,.$$

Therefore $(P_{n-1}/p_n)\,\mu_n\,|\Delta\varepsilon_n| \in$ B.V. and hence $(P_{n-1}/p_n)\,\mu_n\,|\Delta\varepsilon_n| = O(1)$ as $n \to \infty$.

2.3. - Proof of the Theorem.

Let

$$\bar{\sigma}_n^p = (1/P_n) \sum_{v=1}^{n} p_v \sum_{\mu=1}^{v} a_\mu\,\varepsilon_\mu$$

and

$$\bar{t}_n^p = (1/P_n) \sum_{v=1}^{n} P_{v-1}\,a_v\,.$$

Then

$$\bar{\sigma}_n^p - \bar{\sigma}_{n-1}^p = \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) \sum_{v=1}^{n} P_{v-1}\,a_v\,\varepsilon_v$$

$$= \frac{p_n}{P_{n-1}P_n} \sum_{v=1}^{n} P_{v-1}\,a_v\,\varepsilon_v = \frac{p_n}{P_{n-1}P_n}\,S\,,$$

where

$$S = \sum_{v=1}^{n-1} P_v\,\bar{t}_v^p\,\Delta\varepsilon_v + \varepsilon_n P_n\,\bar{t}_n^p = S_1 + S_2\,,$$

say.

Now, as $m \to \infty$, and observing that $\sum_{n=v+1}^{m} \frac{p_n}{P_n P_{n-1}} < \frac{1}{P_v}$, we have

$$\sum_{n=1}^{m} \frac{p_n}{P_{n-1} P_n} |S_1| \leqslant$$

$$\leqslant \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v |\bar{t}_v^p| |\Delta \varepsilon_v| = \sum_{v=1}^{m} P_v |\bar{t}_v^p| |\Delta \varepsilon_v| \sum_{n=v+1}^{m} \frac{p_n}{P_n P_{n-1}}$$

$$= O\Big(\sum_{v=1}^{m} |\bar{t}_v^p| |\Delta \varepsilon_v|\Big) = O\left(\sum_{v=1}^{m} \frac{P_{v-1}}{p_v} |\Delta \varepsilon_v| \frac{p_v}{P_{v-1}} |\bar{t}_v^p|\right)$$

$$= O\left(\sum_{v=1}^{m-1} \frac{P_{v-1}}{p_v} |\Delta^2 \varepsilon_v| \mu_v\right) + O\Big(\sum_{v=1}^{m-1} |\Delta \varepsilon_{v+1}| \mu_v\Big) +$$

$$+ O\Big(\sum_{v=1}^{m-1} P_v |\Delta(1/p_v)| |\Delta \varepsilon_{v+1}| \mu_v\Big) + O\left(\frac{P_{m-1}}{p_m} \mu_m |\Delta \varepsilon_m|\right)$$

$= O(1),$ by hypotheses and Lemma 1 and 2.

Next, as $m \to \infty$, we have

$$\sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}} |S_2| = \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}} |\varepsilon_n| P_n |\bar{t}_n^p|$$

$$= \sum_{n=1}^{m} |\varepsilon_n| \frac{p_n}{P_{n-1}} |\bar{t}_n^p| = O\Big(\sum_{n=1}^{m-1} |\Delta \varepsilon_n| \mu_n\Big) + O(|\varepsilon_m| \mu_m)$$

$= O(1),$ by hypotheses and Lemma 1.

Thus the proof of the Theorem is completed.

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