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CAROL SINGH (*)

An Inversion Integral for a Whittaker Transform. (**)

1. - Introduction.

In [2] K. N. SHRIVASTAVA has given solutions for integral transforms of the type

$$\int_u^1 (t-u)^{\nu-\frac{1}{2}} M_{k,\nu}(t-u)\, g(t)\, \mathrm{d}t = f(u)\,. \tag{1.1}$$

In the present paper we solve the integral equation (1.1) with a different method. It may be of interest to note that the inversion integral we establish does not contain any polynomial or function, secondly it involves a fractional derivative.

The WHITTAKER's functions are defined as

$$W_{k,\mu}(x) = \frac{\Gamma(-2\mu)\ M_{k,\mu}(x)}{\Gamma(\frac{1}{2}-\mu-k)} + \frac{\Gamma(2\mu)\ M_{k,\mu}(x)}{\Gamma(\frac{1}{2}+\mu-k)} \tag{1.2}$$

and

$$M_{k,\mu}(x) = x^{\mu+\frac{1}{2}}\, e^{-x/2}\, {}_1F_1\left(\tfrac{1}{2}+\mu-k;\ 2\mu+1;\ x\right). \tag{1.3}$$

2. – In the result ([1], p. 200, (89)) assuming

(i) $\lambda = -\nu \qquad (0<\nu<\frac{1}{2}),$

(ii) $\mu = k-\nu-\frac{1}{2} \qquad (\mu = 1, 2, \ldots)$

(*) Indirizzo: Department of Mathematics, Government Polytechnic Institute, Raigarh (M. P.), India.

(**) Ricevuto: 3-VI-1969.

we get

$$(2.1)\qquad A\int_0^1 x^{\nu-\frac{1}{2}}\, e^{-ax/2}\, M_{k,\mu}(ax)\,(1-x)^{k-\nu-(3/2)}\,\mathrm{d}x = e^{-a}\, a^{\nu+1/2}\,,$$

where

$$A = \Gamma(-2\nu)/\{\Gamma(\tfrac{1}{2}-\nu-k)\,\Gamma(k-\nu-\tfrac{1}{2})\}.$$

Changing the variable by taking $xa = t-u$ and $a = v-u$, we obtain

$$(2.2)\quad A\int_u^v (t-u)^{\nu-\frac{1}{2}}\, e^{-(t-u)/2}\, M_{k,\nu}(t-u)\,(v-t)^{k-\nu-3/2}\,\mathrm{d}t = e^{-(v-u)}\,(v-u)^{\nu+k-\frac{1}{2}}.$$

$(\mathrm{d}/\mathrm{d}x)^\alpha\, g(x)$ is an ordinary derivative of $g(x)$ if $\alpha = 0, 1, 2, \ldots$; and, if α is not an integer, the fractional derivative is defined by ([1], Chapt. XIII):

$$(2.3)\qquad \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\alpha} g(x) = \frac{1}{\Gamma(n-\alpha)}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^\infty g(s)\,(s-x)^{n-\alpha-1}\,\mathrm{d}s\,,$$

$$n-1<\alpha<n \qquad (n = 0,\ 1,\ 2, \ldots),$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\beta} g(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\alpha+\beta} g(x)\,.$$

3. - Theorem.

If $e^{-u/2} f(u)$ and its first $[\nu+k+\frac{1}{2}]$ derivatives are absolutely continuous, if the $[\nu+k+(3/2)]$ - th derivative is sectionally continuous for $0<u_0\leqslant u\leqslant 1$ and if $e^{-u/2} f(u)$ and its first $[\nu+k+\frac{1}{2}]$ derivatives vanish for $u\geqslant 1$, then the solution to the integral equation (1.1) is given by

$$(3.1)\quad g(t) = \frac{(-1)^{\nu+k+\frac{1}{2}} A}{\Gamma(\nu+k+\frac{1}{2})}\, e^{-t/2}\int_t^1 e^v\,(v-t)^{k-\nu-(3/2)}\left(\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\nu+k+(1/2)}\{e^{-v/2} f(v)\}\,\mathrm{d}v\,.$$

Proof. Substituting the value of $g(t)$ in (1.1) from (3.1), we get

$$(3.2)\qquad I = \frac{(-1)^{\nu+k+\frac{1}{2}}A}{\Gamma(\nu+k+\frac{1}{2})}\int_u^1 (t-u)^{\nu-\frac{1}{2}} M_{k,\nu}(t-u)\cdot$$

$$\cdot\left[e^{-t/2}\int_t^1 e^v (v-t)^{k-\nu-(3/2)}\left(\frac{d}{dv}\right)^{\nu+k+\frac{1}{2}}\{e^{-v/2}f(v)\}\,dv\right]dt\,.$$

The change of the order of integration gives us

$$(3.3)\qquad I = \frac{(-1)^{\nu+k+\frac{1}{2}}A}{\Gamma(\nu+k+\frac{1}{2})}\int_u^1 e^v\left(\frac{d}{dv}\right)^{\nu+k+\frac{1}{2}}\{e^{-v/2}f(v)\}\cdot$$

$$\cdot\Big[\int_u^v (t-u)^{\nu-\frac{1}{2}} M_{k,\nu}(t-u)\, e^{-t/2}(v-t)^{k-\nu-(3/2)}\,dt\Big]\,dv\,.$$

Using the result (2.2) we get from (3.3):

$$(3.4)\qquad I = \frac{(-1)^{\nu+k+\frac{1}{2}}}{\Gamma(\nu+k+\frac{1}{2})}\int_u^1 e^{u/2}(v-u)^{\nu+k+\frac{1}{2}}\left(\frac{d}{dv}\right)^{\nu+k+\frac{1}{2}}\{e^{-v/2}f(v)\}\,dv\,.$$

Since $0<\nu<\frac{1}{2}$ and $\mu=k-\nu-\frac{1}{2}$ ($\mu=1,2,\ldots$), we get $\nu+k+\frac{1}{2}=\mu+$ $+2\nu+1$ and also $\mu+2\nu+1>2$. Obviously $\mu+2\nu+1$ is not an integer. The relation (3.4) can be rewritten as

$$(3.5)\qquad I = \frac{(-1)^{\mu+1}e^{u/2}}{\Gamma(\mu+1+2\nu)}\int_u^1 (v-u)^{\mu+2\nu}\left(\frac{d}{dv}\right)^{\mu+2\nu+1}\{e^{-v/2}f(v)\}\,dv\,.$$

Integration by parts for $\mu+1$ times under the conditions of the theorem, gives

$$(3.6)\qquad I = \frac{e^{u/2}}{\Gamma(2\nu)}\int_u^1 (v-u)^{2\nu-1}\left(\frac{d}{dv}\right)^{2\nu}\{e^{-v/2}f(v)\}\,dv\,.$$

But (2.3) with $n=0$, $\alpha=-2\nu$, $x=u$ and $g(s)=0$ for $s\geqslant 1$ becomes

$$\left(\frac{d}{du}\right)^{-2\nu} g(u) = \frac{1}{\Gamma(2\nu)}\int_u^1 g(s)\,(s-u)^{2\nu-1}\,ds$$

from which it follows immediately that

$$I = e^{u/2} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{-2r} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{2r} \left\{e^{-u/2} f(u)\right\} \qquad \text{or} \qquad I = f(u) .$$

This establishes the Theorem.

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References.

[1] A. ERDÉLYI, *Tables of Integral Transforms*, Vol. 2, McGraw-Hill, New York 1954.

[2] K. N. SRIVASTAVA, *On integral equations involving Whittaker's function*. Proc. Glasgow Math. Assoc. **7** (1966), 125-127.

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