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The Degree Preserving Group of a Graph. (**)

1. - Introduction and Definitions.

We consider (ordinary) graphs, that is, finite undirected graphs with no loops or multiple edges. The (vertex) *group* $\Gamma(G)$ *of a graph* G is the group of all adjacency preserving permutations of the vertices of G. It is clear from the definition that if $\varphi \in \Gamma(G)$, and $v \in V(G)$, then $\deg(\varphi v) = \deg v$, where $V(G)$ denotes the vertex set of G and $\deg v$ denotes the degree of v. Hence $\Gamma(G)$ is a subgroup of $\Gamma_d(G)$, where $\Gamma_d(G)$ called *the degree preserving group of* G denotes the group of permutations of the elements of $V(G)$ such that, for each $v \in V(G)$ and $\varphi \in \Gamma_d(G)$, we have: $\deg v = \deg(\varphi v)$. The group $\Gamma(G)$ as well as some other groups associated with a graph G and their relations have been the subject of many investigations. (See, for example, [1] [2], [3] and [4].) In this Note we find necessary and sufficient conditions to have $\Gamma(G) \cong \Gamma_d(G)$. (In fact, we can replace the isomorphism sign $\cong$ by the identity sign $\equiv$.)

2. - Results.

In the first theorem we consider disconnected graphs, that is, graphs with two or more components.

THEOREM 1. *Let* G *be a disconnected graph. Then* $\Gamma(G) \cong \Gamma_d(G)$ *if and only if:*

(i) *For every component* H *of* G *we have* $\Gamma(H) \cong \Gamma_d(H)$, *and*

(ii) *no two nontrivial components of* G *have vertices of the same degree.*

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Proof. The necessity of (i) is obvious. If two nontrivial components of G, say H_1 and H_2 contain vertices v_1 and v_2, respectively, such that $\deg v_1 = \deg v_2$, then we can find an element in $\Gamma_d(G)$ which is not a member of $\Gamma(G)$. Hence the condition (ii) is necessary as well.

To prove the sufficiency of the conditions it is sufficient to show that an arbitrary element α of $\Gamma_d(G)$ is also an element of $\Gamma(G)$. By (ii) α maps vertices of a nontrivial component H of G to itself; thus (i) implies that the restriction of α to $V(H)$ preserves adjacency. This completes the proof of the theorem.

According to Theorem 1, we must confine ourselves to connected graphs. The *neighborhood* $N_G(v)$ *of a vertex* v of a graph G is defined to be the set of all vertices of G which are adjacent to v; the *closed neighborhood* $\overline{N}_G(v)$ of v is $N_G(v) \cup \{v\}$. (Two vertices which are not adjacent are *disjoint*.)

Theorem 2. *Let G be a connected graph. Then* $\Gamma(G) \cong \Gamma_d(G)$ *if and only if:*

(i) *Equidegree vertices of G two of which are disjoint are all mutually disjoint and all have the same neighborhood, and*

(ii) *equidegree vertices of G two of which are adjacent are all mutually adjacent and all have the same closed neighborhood.*

Proof. We first assume that $\Gamma(G) \cong \Gamma_d(G)$. Let S be the set of all elements of $V(G)$ each of which has degree d. If $|S| = 1$, then there is nothing to prove. Hence, suppose that $|S| \geqslant 2$.

Let $u_1, u_2 \in S$ such that $u_1u_2 \notin E(G)$, where $E(G)$ denotes the edge set of G, and let $v_1, v_2 \in S$ such that $v_1 v_2 \in E(G)$. If $\{u_1, u_2\} \cap \{v_1, v_2\} = \phi$, then we define a permutation α on $V(G)$ as follows:

$$\alpha u_1 = v_1, \qquad \alpha v_1 = u_1,$$
$$\alpha u_2 = v_2, \qquad \alpha v_2 = u_2$$

and

$$\alpha w = w,$$

where $w \in V(G) - \{u_1, u_2, v_1, v_2\}$. It is easily seen that $\alpha \in \Gamma_d(G)$ and $\alpha \notin \Gamma(G)$, which is a contradiction. If $\{u_1, u_2\} \cap \{v_1, v_2\} \neq \phi$, we might suppose that $v_1 = u_1$. Again, it is easy to sohw that $\Gamma_d(G)$ contains elements which are not in $\Gamma(G)$. Thus in any case, if two elements of S are adjacent, then all elements of S are mutually adjacent; and conversely, if a pair of elements of S are disjoint, then all elements of S are mutually disjoint.

Next, we suppose that elements of S are mutually disjoint and prove that they must have the same neighborhood. Assume to the contrary that S contains two elements u_1 and u_2 with $N_G(u_1) \neq N_G(u_2)$. Then we define a permutation α on $V(G)$ as follows:

$$\alpha u_1 = u_2 , \qquad \alpha u_2 = u_1$$

and

$$\alpha w = w ,$$

where $w \in V(G) - \{u_1 , u_2\}$. Then $\alpha \in \Gamma_d(G)$, and $\alpha \notin \Gamma(G)$ produce a contradiction.

Following the above argument we conclude that if elements of S are all mutually adjacent, then they must have the same closed neighborhood. This completes the proof of the necessity of the conditions.

To prove the sufficiency, we must show that $\Gamma_d(G) \subset \Gamma(G)$; i.e., we must show that each element α of $\Gamma_d(G)$ preserves adjacency. Let $u, v \in V(G)$ such that $u\,v \in E(G)$. To show that $\alpha\,u\;\alpha\,v \in E(G)$, we consider two cases.

Case 1. $\deg u = \deg v = d$. Since $\deg(\alpha\,u) = \deg(\alpha\,v) = d$, we conclude from (ii) that $\alpha\,u\;\alpha\,v \in E(G)$.

Case 2. $\deg u \neq \deg v$. Since $\deg(\alpha\,u) = \deg u$, and $v \in N_G(u)$ we have: $v \in N_G(\alpha\,u)$. Thus $v\,\alpha\,u \in E(G)$ which implies that $\alpha\,u \in N_G(v)$. Thus $\alpha\,u \in N_G(\alpha\,v)$. Hence $\alpha\,u\,\alpha\,v \in E(G)$, as was required to prove.

Corollary 1. *Among regular graphs G, the complete graph and its complement are the only graphs for which $\Gamma(G) \cong \Gamma_d(G)$.*

Corollary 2. *Among trees having two or more vertices, the star graph $K_{1,n}$, $n \geqslant 1$, is the only tree for which we have $\Gamma(K_{1,n}) \cong \Gamma_d(K_{1,n})$.*

Proof. Let T be a tree whose diameter is greater than two with the property that $\Gamma(T) \cong \Gamma_d(T)$. Then T contains two nonadjacent vertices of degree 1. By Theorem 2 this is impossible.

Corollary 3. *Let G be a graph such that $\Gamma(G) \cong \Gamma_d(G)$ and let v be a cut-vertex of G. Then the degree of no other vertex of G equals that of* $\deg v$.

Proof. Assume to the contrary that G contains a vertex v', $v' \neq v$, such that $\deg v = \deg v'$. Then by Theorem 1, v and v' belong to the same component H of G. Suppose u and w are two vertices of H adjacent to v which lie in two different components of $H - v$. If $v' \in \{u, w\}$, that is, if, say $v' = u$, then $uw \in E(G)$ contradicts the fact that v is a cut-vertex of G. Hence, we assume that $v' \notin \{u, w\}$. Then v' is adjacent to both u and w and again we contradict the fact that v is a cut-vertex of G. This completes the proof.

It might be conjectured that if G is a graph with a cut-vertex v, and such that $\Gamma(G) \cong \Gamma_d(G)$, then:

(1) Every block B of G has the property $\Gamma(B) \cong \Gamma_d(B)$, and
(2) every component H of $G-v$ has the property $\Gamma(H) = \Gamma_d(H)$.

It is easy to find counter examples to show that neither (1) nor (2) is, in general, correct. In fact, the graph given in Fig. 1 is a counter example for both.

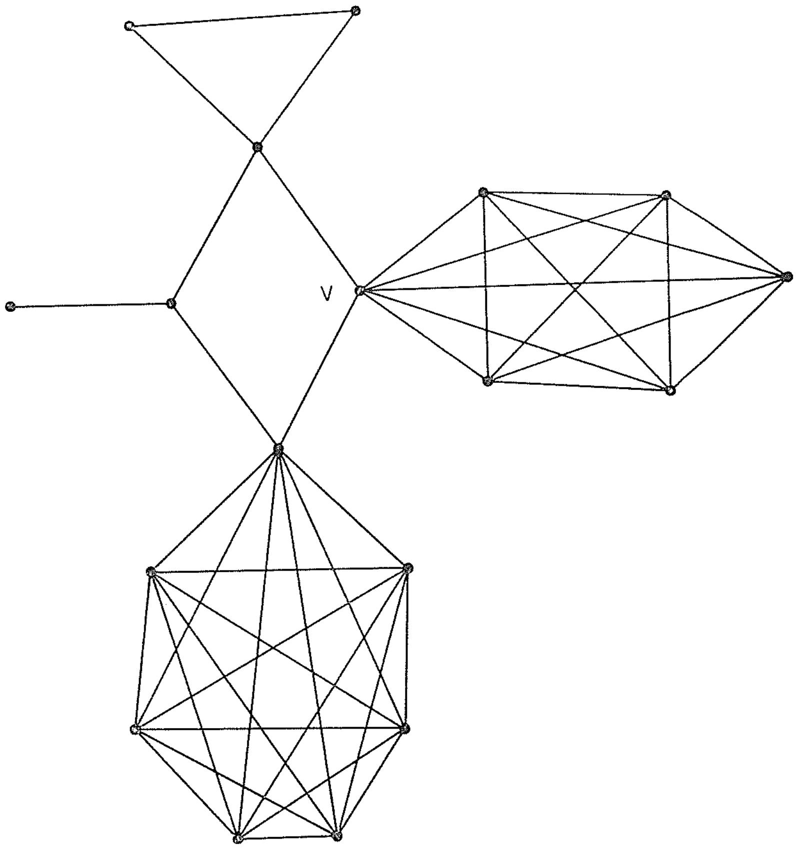

Fig. 1.

It is also worth mentioning that if G is a graph with a cut-vertex v and if every component H of $G-v$ has the property $\Gamma(H) \cong \Gamma_d(H)$, then the graph G

might not have the property $\Gamma(G) \cong \Gamma_d(G)$. On the other hand, if every block B of G has the property $\Gamma(B) \cong \Gamma_d(B)$, then not necessarily the relation $\Gamma(G) \cong \Gamma_d(G)$ is true.

References.

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Abstract.

Though the (vertex) group $\Gamma(G)$ of a graph G is fundamental in graphical enumerations, yet there is no practical method for producing groups of graphs. In this Note we define the degree preserving group $\Gamma_d(G)$ of a graph G as the group of all permutations of the vertices of G each of which preserves the degree of each vertex of G, and characterize those graphs G for which $\Gamma(G)$ and $\Gamma_d(G)$ are isomorphic. Thus we yield an easy method for producing the group of a certain class of graphs.

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