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An Integral Equation with Jacobi Polynomial in the Kernel. (**)

1. – In [4] K. N. SRIVASTAVA has solved the integral equations

$$\int_x^1 F_n^{(k,\beta)}(t/x)\, g(t)\, \mathrm{d}t = f(x)\,, \tag{1.1}$$

$$\int_x^1 R_n^{(k,\beta)}(t/x)\, g_1(t)\, \mathrm{d}t = f_1(x)\,, \tag{1.2}$$

where

$$F_n^{(k,\beta)}(x) = \frac{n!\, x\, (x^2-1)^{k-\beta}\, p_n^{(k-\beta,\beta)}(2x^2-1)}{2^{k-\frac{1}{2}}\, \Gamma(k-\beta+n+1)}\,,$$

$$R_n^{(k,\beta)}(x) = \frac{n!\, (x^2-1)^{k+\beta-1}\, p_n^{(k+\beta-1),(-\beta)}(2x^2-1)}{2^{k-\frac{1}{2}}\, \Gamma(k+\beta+n)}\,.$$

Here $p_n^{(A,B)}(x)$ stand for the JACOBI polynomial. SRIVASTAVA has used the method of R. G. BUSCHMAN [1] which involves the use of MELLIN-transform. In the present paper we solve a similar integral equation whose Kernel is the JACOBI polynomial itself. It may be of interest to note that the Kernel we use is in quite a simple form and also its argument is different than that of $F_n^{(k,\beta)}(x)$ or $R_n^{(k,\beta)}(x)$. We shall use a different method which is more direct and leads to the solution under fewer conditions on $f(x)$.

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2. – Consider the integral equation

$$\int_1^x p_n^{(\alpha,\beta)}(1-(2x/t))\, g(t)\, dt = f(x)\,, \qquad 1 < x < x_0\,. \tag{2.1}$$

If $n \geqslant 1$, $f(x)$ is absolutely continuous on $[1, x_0]$ for some $x_0 > 1$ and $f(1) = 0$, then the solution of (2.1) is given by

$$g(x) = \frac{A\, x^n}{(\beta+n-1)!\; n!}\left(\frac{d}{dx}\right)^{n+1}\left\{x^{-(\alpha+\beta+n)}\int_1^x (x-t)^{\beta+n-1}\, t^\alpha\, f(t)\, dt\right\}, \tag{2.2}$$

where

$$A = n!\,(1+\alpha)_{\beta+n}/(1+\alpha)_n\,.$$

From ([3], p. 99) we have the relation

$$\left(\frac{d}{dx}\right)^m\left\{x^{b+m-1}\,(1-x)^{a+m-b}\right\} = b_m\, x^{b-1}\,{}_2F_1(b-a-m\,,\; b+m;\; b;\; x)\,.$$

The above relation can be easily transformed in the following form

$$\left(\frac{d}{dx}\right)^{\beta+n}\left\{x^{\alpha+\beta+n}\,(1-x)^n\right\} = A\, x^\alpha\, p_n^{(\alpha,\beta)}(1-2x)\,, \tag{2.3}$$

where

$$A = n!\,(1+\alpha)_{\beta+n}/(1+\alpha)_n\,.$$

In the light of the result (2.3) the integral equation (2.1) can be rewritten as

$$\int_1^x \left(\frac{d}{dx}\right)^{\beta+n}\left\{x^{\alpha+\beta+n}\,(t-x)^n\right\} t^{-n}\, g(t)\, dt = A\, f(x)\, x^\alpha\,.$$

Interchanging the operators of integration and differentiation, which is justified, here we get

$$\left(\frac{d}{dx}\right)^{\beta+n}\int_1^x \left\{x^{\alpha+\beta+n}\,(t-x)^n\right\} t^{-n}\, g(t)\, dt = A\; f(x)\, x^\alpha\,. \tag{2.4}$$

Since the left hand side of (2.4) vanishes together with its $\beta + n - 1$ derivatives when $x = 1$, repeated integration for $\beta + n$ times gives us

$$(2.5) \qquad \int_1^x (t - x)^n t^{-n} g(t)\, dt = \frac{A\, x^{-(\alpha+\beta+n)}}{(\beta + n - 1)!} \int_1^x (x - t)^{\beta+n-1} t^{\alpha} f(t)\, dt\,.$$

Finally differentiating for $n + 1$ times with respect to x, we get (2.2).

3. - The integral equation

$$(3.1) \qquad \int_x^1 (x/t)^{\nu}\, p_n^{(\alpha,\beta)}(1 - (2x/t))\, g_2(t)\, dt = f_2(x)\,, \qquad 1 < x < x_0\,,$$

is reduced to (2.1) if $\nu = 0$. However, if we write $t^{-\nu} g_2(t) = g(t)$ and $t^{-\nu} f_2(t) = f(t)$, the equation (3.1) is again reduced to (2.1), hence we get the solution of (3.1) as

$$(3.2) \quad g_2(x) = \frac{A\, x^{\nu+n}}{(\beta + n - 1)!\; n!} \left(\frac{d}{dx}\right)^{n+1} \left\{ x^{-(\alpha+\beta+n)} \int_1^x (x - t)^{\beta+n-1} \left[t^{\alpha-\nu} f_2(t)\right] dt \right\}.$$

If $\alpha = \beta = 0$, is put in (2.1) and (2.2) we get the solution of the integral equation

$$(3.3) \qquad \int_x^1 p_n(1 - (2x/t))\, g(t)\, dt = f(x)\,, \qquad 1 < x < x_0\,,$$

as

$$(3.4) \qquad g(x) = \frac{1}{(n-1)!} \left(\frac{d}{dx}\right)^{n+1} \left\{ x^{-n} \int_1^x (x - t)^{n-1} f(t)\, dt \right\}.$$

The integral equation (3.3) is different than the one solved by ERDELYI [2].

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References.

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[4] K. N. SRIVASTAVA, *Inversion integrals involving Jacobi polynomial*, Proc. Amer. Math. Soc. **15** (1963), 634-638.

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