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MARTHA M. MATTAMAL (*)

Axiomatic Characterization of the Lebesgue Integral. (**)

In this paper we define the LEBESGUE integral axiomatically as a countably additive positive linear functional on a linear lattice of real valued functions satisfying the STONE condition over an abstract set. Considering only the real valued functions in some of the basic constructions of the integrals existing in the literature, we show briefly that the integrals thus constructed satisfy the axioms. The integrals of LEBESGUE, RIESZ, DANIELL, STONE, BOURBAKI, the integral generated by a measure and the integral generated by a volume are shown to be such examples of the LEBESGUE integral.

The main result of the paper is in Section 4 where we show that the LEBESGUE integral may be generated by a volume (as defined by BOGDANOWICZ in [1]) or by a measure.

§ 1. - Axiomatic definition of a Lebesgue integral.

We shall denote the non-negative subfamily of a given family F of real valued functions by F^+. For the functions f, g defined on a set X we shall define the operations $f \cup g$, $f \cap g$, $f \cap 1$, $|f|$ by

$$(f \cup g)(x) = \sup\{f(x), g(x)\}, \qquad (f \cap g)(x) = \inf\{f(x), g(x)\},$$

$$(f \cap 1)(x) = \inf\{f(x), 1\}, \qquad |f|(x) = |f(x)| \qquad \text{for all } x \in X.$$

(*) Indirizzo: Department of Mathematics, Howard University, Washington, D.C., 20001, U.S.A..

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Definition. By a LEBESGUE integral we shall mean a real valued functional $\int$ with domain $D(\int) = L$ consisting of functions from a set X into the space R of reals and satisfying the following conditions:

(1) L is a linear space and $\int$ is a linear functional on L; that is if $f, g \in L$ and $a, b \in R$, then $af + bg \in L$ and $\int(af + bg) = a\int f + b\int g$.

(2) If $f, g \in L$, then $f \cup g \in L$.

(3) L satisfies the STONE condition; that is if $f \in L$ then $f \cap 1 \in L$.

(4) $\int$ is a positive functional on L; that is if $f \in L^+$ then $\int f \geqslant 0$.

(5) $\int$ is a countably additive functional on L; that is if $f_n \in L^+$, $f(x) = \sum_{n=1}^{\infty} f_n(x) < \infty$ for all $x \in X$ and $\sum_{n=1}^{\infty} \int f_n < \infty$, then $f \in L^+$ and $\int f = \sum_{n=1}^{\infty} \int f_n$.

The LEBESGUE integral is said to be complete if L contains every non-negative function g on X to which there corresponds an $f \in L$ such that $0 \leqslant g(x) \leqslant f(x)$ for all $x \in X$ and $\int f = 0$.

Remark 1. The second condition in the definition can be replaced by (2)′ $f \cap g \in L$ for all $f, g \in L$ (so that L is a lattice), or by (2)″ $|f| \in L$ for all $f \in L$.

Remark 2. Condition (5) in the definition can be replaced by:

(5)′ If $g_n \in L$ is an increasing sequence convergent at every point of the set X to a finite valued function g and the sequence $\int g_n$ is bounded, then $g \in L$ and $\int g_n$ converges to $\int g$.

§ 2. - Examples of the Lebesgue integral.

In the following examples of integrals of real valued functions we use in each case the terminology and notation of the literature referred to.

Example 1. The integral generated by a measure on an abstract set X (see [7]) is a LEBESGUE integral. Axioms (1), (2), (4), (5) are well known properties of the class of summable functions. To prove (3) we first note that, if f is measurable, then so is $f \cap 1$. The summability of $f \cap 1$ now follows from the inequality $|f \cap 1| \leqslant |f|$.

If the measure μ is complete, then we shall show that the integral is a complete LEBESGUE integral. Let $0 \leqslant g(x) \leqslant f(x)$ for all $x \in X$ and $\int f d\mu = 0$. Then $f(x) = 0$ almost everywhere and thus $g(x) = 0$ except on a subset of a set of measure zero. Since the measure is complete, g is measurable and $g(x) = 0$ almost everywhere. Hence g is summable.

This example also covers the case of the classical LEBESGUE integral on the real line. In [8] it is shown that the integral of LEBESGUE coincides with that of RIESZ.

Example 2. If the initial class T_0 considered by DANIELL [6] satisfies the STONE condition, then the integral developed by DANIELL is a complete LEBESGUE integral in our sense. We shall assume that $f \cap 1 \in T_0$ for all $f \in T_0$.

It is established in Section 7 of [6] that the space of summable functions is a linear lattice on which I is a positive linear countably additive functional.

To prove the completeness, let $0 \leqslant g \leqslant f$ and $I(f) = 0$. Since $\dot{I}(g) \leqslant \dot{I}(f)$ and $\underset{\cdot}{I}(g) \leqslant \underset{\cdot}{I}(f)$, we see that $\dot{I}(g) = \underset{\cdot}{I}(g) = 0$, and hence g is summable.

We shall prove that the space of summable functions satisfies the STONE condition. Take any summable function f. Since $\dot{I}(f) = \underset{\cdot}{I}(f)$, for any $\varepsilon > 0$ there exist $h, k \in T_1$ such that $-k \leqslant f \leqslant h$ and $I(h + k) < \varepsilon$. Then $h \cap 1 \in T_1$, $k \cup (-1) \in T_1$, and

$$-(k \cup -1) \leqslant f \cap 1 \leqslant h \cap 1 .$$

Moreover, since $h + k \geqslant 0$,

$$I[(h \cap 1) + (k \cup -1)] \leqslant I(h + k) < \varepsilon .$$

It follows that $\dot{I}(f \cap 1) = \underset{\cdot}{I}(f \cap 1)$, i.e. that $f \cap 1$ is summable.

Example 3. The integral constructed by STONE [9] is a complete LEBESGUE integral if the initial class satisfies the STONE condition.

It is established if [9]I that the class of integrable functions is a linear lattice and L is a countably additive positive linear functional on it. In [9]II is shown that if the initial class satisfies the STONE condition then the same is true of the class of integrable functions.

To prove the completeness, let $0 \leqslant g \leqslant f$ and $L(f)0$. Since $f(x)$ is positive for all x, $L(f) = F(f) = N(f)$. Since $g \leqslant f$ we have $N(g) \leqslant N(f)$ which implies that $F(g) = N(g) = 0$. Thus g is integrable.

Example 4. The integral constructed by BOURBAKI [5] on a locally compact space is a complete LEBESGUE integral when the functions are real valued.

It is easy to see from the theory that the space of integrable functions is a linear lattice on which the integral is a linear functional. The positivity of the integral follows from Proposition 1 of § 4 and the countable additivity from Theorem 5 of § **3** of Chapter IV. The STONE condition follows by applying Theorem 3 and Proposition 5 of § **4** of Chapter IV to f and $f \cap 1$. Proposition 1 of § **4** of Chapter IV yields the completeness of the integral.

Example 5. The integral generated by a volume v (as in [**1**]) is a complete LEBESGUE integral. For a summable function f we shall show that $f \cap 1$ is summable. There exists a basic sequence s_n such that $s_n(x)$ converges to $f(x)$ v-almost everywhere. Then the sequence $s_n \cap 1$ is basic, because

$$\|s_{n+1} \cap 1 - s_n \cap 1\| \leqslant \|s_{n+1} - s_n\| .$$

Since $s_n \cap 1$ converges to $f \cap 1$ v-a.e., $f \cap 1$ is summable. The other axioms of a complete LEBESGUE integral are obviously satisfied.

§ 3. - The volume generated by a Lebesgue integral.

A family V of subsets of an abstract space X is called a prering if for every two sets $A, B \in V$ the sets $A \cap B$ and $A \setminus B$ can be represented as finite unions of disjoint sets from the family V.

A non-negative function v defined on the prering V is called a volume if for every countable family of disjoint sets $A_n \in V$ such that $A = \bigcup_{n=1}^{\infty} A_n \in V$, we have $v(A) = \sum_{n=1}^{\infty} v(A_n)$. The triple (X, V, v) is called a volume space.

A volume v with domain V is called an upper complete volume if the following conditions are satisfied:

(1) The family V is a ring, that is it is a prering satisfying the condition that if $A, B \in V$, then $A \cup B \in V$.

(2) For every increasing sequence of sets $A_n \in V$ such that the sequence $v(A_n)$ is bounded, we have $\bigcup_{n=1}^{\infty} A_n \in V$.

If in addition the pair of conditions $A \subset B \in V$ and $v(B) = 0$ implies $A \in V$, then the volume is said to be complete.

It is easy to see that a volume v defined on a prering V is upper complete if and only if the following condition is satisfied:

If $A_n \in V$ is a sequence of disjoint sets such that $\sum_{n=1}^{\infty} v(A_n) < \infty$, then $\bigcup_{n=1}^{\infty} A_n \in V$.

Theorem 1. *If* $\int$ *is a Lebesgue integral over* X, *then* $V = \{A \subset X: c_A \in D(\int)\}$, *where* c_A *is the characteristic function of the set* A, *is a ring and the function* v *defined on* V *by* $v(A) = \int c_A$ *for* $A \in V$ *is an upper complete volume. When the integral is complete, the volume is complete.*

Proof. Let $A, B \in V$. The domain $L = D(\int)$ of the integral being a linear lattice, the following is true:

$$c_{A\cup B} = c_A \cup c_B \in L\,, \qquad c_{A\cap B} = c_A \cap c_B \in L$$

and

$$c_{A\setminus B} = c_A - c_{A\cap B} \in L\,.$$

Thus V is a ring.

From the definition of the function v it is clear that it is finite valued and non-negative. Now let $A, A_n \in V$, A_n disjoint and $A = \bigcup_{n=1}^{\infty} A_n$. Then $c_{A_n}, c_A \in L^+$, and

$$\sum_{n=1}^{\infty} c_{A_n}(x) = c_A(x) < \infty \qquad \text{for all } x \in X\,.$$

Also,

$$\sum_{n=1}^{k} \int c_{A_n} = \int \sum_{n=1}^{k} c_{A_n} \leqslant \int c_A < \infty \qquad \text{for all } k\,.$$

This implies by the countable additivity of the integral that $v(A) = \sum_{n=1}^{\infty} v(A_n)$. Thus v is a volume. The countable additivity of the integral also implies the upper completeness of the volume. When the integral is complete, the volume is clearly complete. This proves the theorem.

We shall call the family V of the above theorem, the family of summable sets generated by the integral and v the volume generated by the integral.

Theorem 2. *If* $\int$ *is a Lebesgue integral over* X, *then for every function* $f \in L = D(\int)$ *and every* $a > 0$ *the set* $A_a = \{x \in X: f(x) > a\}$ *is summable.*

Proof. Let $f \in L$. The sequence of functions f_n given by

$$f_n = n(f - f \cap a) \cap 1$$

converges increasingly to the function c_{A_a}. Since

$$f \cap a = a(a^{-1} f \cap 1) \in L\,,$$

the function $f_n \in L$.

Now,

$$A_a = \{x \in X:\ f^+(x) > a\}$$

where f^+ is the positive part of the function f and therefore

$$a^{-1} f^+(x) \geqslant c_{A_a}(x) \qquad \text{for all } x \in X\,.$$

Since $f_n \leqslant c_{A_a} \leqslant a^{-1} f^+$, the integral of f_n is bounded and, by the countable additivity of the integral $c_{A_a} \in L$. The theorem is therefore proved.

For a given LEBESGUE integral $\int$ over X, the subset A of X is called a null set generated by the integral if $\int c_A = 0$. It is clear that the union of a countable family of null sets is a null set and, when the integral is complete, any subset of a null set is itself a null set.

A condition $C(x)$ depending on the parameter x is said to be satisfied $\int$-almost everywhere ($\int$-a.e.) on X if $C(x)$ holds for all $x \notin A$ where A is a null set generated by the integral.

Theorem 3. *Let $\int$ be a complete Lebesgue integral over X and f a finite valued function on X such that $f(x) = 0$ $\int$-a.e. Then $f \in D(\int)$ and $\int f = 0$.*

Proof. We may assume that f is non-negative. Then $f(x) = 0$ for $x \notin A$, where $A \in V_0$ and V_0 is the family of null sets generated by the integral. For $a > 0$, the set

$$A_a = \{x \in X:\ f(x) > a\}$$

is contained in A and, since the integral is complete, $A_a \in V_0$.

Define a sequence of functions s_n by

$$s_n = \sum_j 2^{-n} j\, c_{B_{nj}} \qquad (j = 1, \ldots, 4^n)$$

where

$$B_{nj} = \{x \in X:\ 2^{-n} j < f(x) \leqslant 2^{-n}(j+1)\}\,.$$

Then

$$B_{nj} \subset A_{2^{-n}j} \in V_0$$

and therefore $B_{nj} \in V_0$, that is

$$c_{B_{nj}} \in D(\textstyle\int) \quad \text{and} \quad \int c_{B_{nj}} = 0\,.$$

This implies that

$$s_n \in D(\textstyle\int) \quad \text{and} \quad \int s_n = 0\,.$$

But $s_n(x)$ converges increasingly to $f(x)$ for all $x \in X$. Hence by the countable additivity of the integral $f \in D(\int)$ and $\int f = 0$.

§ 4. - Representation of Lebesgue integrals.

In [1] is presented an approach to the theory of the space $L(v, Y)$ of BANACH-space valued LEBESGUE-BOCHNER summable functions generated by a volume space (X, V, v). Here we shall consider the case of the space $L(v, R)$ of real valued LEBESGUE summable functions.

The family $S(V, R)$ of simple functions over the prering V is the family of all functions f of the form

$$f = r_1 c_{A_1} + \ldots + r_k c_{A_k} ,$$

where $r_i \in R$, $A_i \in V$ and the A_i are mutually disjoint. The integral of f is defined by

$$\int f \, dv = r_1 v(A_1) + \ldots + r_k v(A_k) .$$

The functional on $S(V, R)$ given by

$$\|f\|_v = \int |f| \, dv$$

is a semi-norm on $S(V, R)$.

The subset A of X is called a v-null set if, given $\varepsilon > 0$, there exists a countable collection of sets $A_n \in V$ such that

$$A \subset \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} v(A_n) < \varepsilon .$$

A condition $C(x)$ depending on a parameter $x \in X$ is said to be satisfied v-almost everywhere (v-a.e.) if there exists a v-null set A such that the condition is satisfied at every point of the set $X \setminus A$.

A sequence of functions $s_n \in S(V, R)$ is called a basic sequence if there exists a constant $M > 0$ such that, for $n = 1, 2, \ldots$, $\|s_n - s_{n-1}\| \leqslant M 4^{-n}$ (where $s_0 \equiv 0$). The space $L(v, R)$ of LEBESGUE summable functions is the set of all functions f for which there exists a basic sequence s_n convergent v-almost everywhere to the function f. For such f we define

$$\|f\|_v = \lim \|s_n\|_v \qquad \text{and} \qquad \int f \, dv = \lim \int s_n \, dv .$$

The space $L(v, R)$ is a linear space and $\|f\|_v$ is a semi-norm on $L(v, R)$. The functional $\int f dv$ is positive and linear on $L(v, R)$. For details we refer the reader to [1].

Lemma 1. *Let $\int$ be a Lebesgue integral and v the volume generated by $\int$ on the prering V of summable sets. If $h \in S(V, R)$, then $h \in D(\int)$ and $h \in L(v, R)$. Moreover, $\int h = \int h \, dv$.*

Proof. $S(V, R)$ being the set of simple functions of the prering V, is contained in the space $L(v, R)$.

Let $h = r_1 c_{A_1} + \ldots + r_k c_{A_k}$, where $r_i \in R$ and $A_i \in V$. By Theorem 1, $h \in D(\int)$ and

$$\int h = r_1 v(A_1) + \ldots + r_k v(A_k) = \int h \, dv .$$

Lemma 2. *Let $\int$ be a Lebesgue integral over X and v the volume generated by $\int$ on the prering V of summable sets. Let J be a complete Lebesgue integral over X such that $\int \subset J$, that is $D(\int) \subset D(J)$ and $\int f = Jf$ for all $f \in D(\int)$. Let w be the volume generated by J on the prering W of summable sets generated by J. Then $N_v \subset W_0$, where N_v is the family of v-null sets and W_0 the family of null sets generated by the integral J.*

Proof. Let $A \in N_v$. Then for $\varepsilon > 0$ there exists a sequence $A_n \in V$ such that

$$A \subset \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \sum_{n=1}^{\infty} v(A_n) < \varepsilon .$$

$A_n \in V$ implies

$$c_{A_n} \subset D(\int) \quad \text{and} \quad v(A_n) = \int c_{A_n} .$$

But since $\int \subset J$, this implies that $A_n \in W$ and $w(A_n) = v(A_n)$.

The sequence of sets $C_n = A_1 \cup \ldots \cup A_n$ is increasing and C_n belongs to the ring W. Moreover,

$$W(C_n) \leqslant w(A_1) + \ldots + w(A_n) < \varepsilon$$

and therefore by the upper completeness of w we get

$$B_\varepsilon = \bigcup_{n=1}^{\infty} A_n \in W .$$

also

$$w(B_\varepsilon) \leqslant \sum_{n=1}^{\infty} w(A_n) < \varepsilon .$$

Now, if we set $\varepsilon = 1/n$, then there exists $B_n \in W$ such that $A \subset B_n$ and $w(B_n) < 1/n$. It is clear from the upper completeness of w that the set $B = \bigcap_{n=1}^{\infty} B_n$ is a member of W. Moreover,

$$0 \leqslant w(B) \leqslant w(B_n) \leqslant 1/n \qquad \text{for all } n$$

which implies $w(B) = 0$. We have $A \subset B$ where $w(B) = 0$ and therefore by the completeness of w, $A \in W$ and $w(A) = 0$. This proves that $N_v \subset W_0$.

Lemma 3. *Let $\int$ be a complete Lebesgue integral and v the volume generated by the integral. Then the family of v-null sets coincides with the family of null sets generated by $\int$.*

This Lemma is an obvious consequence of Lemma 2.

Theorem 4. *Let $\int$ be a complete Lebesgue integral over X and v the volume generated by $\int$. Then $D(\int) = L(v, R)$ and $\int f = \int f \mathrm{d}v$ for all $f \in D(\int)$.*

Proof. We shall denote $D(\int)$ by L. The desired result will follow if we prove that $L^+ = L^+(v, R)$ and $\int f = \int f \mathrm{d}v$ for all $f \in L^+$.

Take $f \in L^+$ and consider the sequence of functions

$$s_n = \sum_j 2^{-n} j \, c_{B_{nj}} \qquad (j = 1, \ldots, 4^n)$$

where

$$B_{nj} = \{x \in X\colon\ 2^{-n} j < f(x) \leqslant 2^{-n}(j+1)\}.$$

Let V be the ring of summable sets on which the volume v is defined. By Theorem 2, the set

$$\{x \in X\colon\ 2^{-n} j < f(x)\}$$

belongs to V and hence the set B_{nj} belongs to V. This implies that $s_n \in S^+(V, R)$ and therefore, by Lemma 1, $s_n \in L^+$ and $\int s_n = \int s_n \mathrm{d}v$.

The sequence s_n increasingly converges everywhere on X to $f \in L^+$ and $\int s_n \leqslant \int f$ for all n which implies, by the countable additivity of the integral $\int$, that $\int s_n$ converges to $\int f$. Since $s_n \in L^+(v, R)$ and $\int s_n \mathrm{d}v \leqslant \int f$, we get from theorems 4(d) and 1(2) of [**1**] that $f \in L^+(v, R)$ and

$$\int f \, \mathrm{d}v = \lim \int s_n \, \mathrm{d}v = \lim \int s_n .$$

Thus we have proven that $L^+ \subset L^+(v, R)$ and $\int f = \int f \mathrm{d}v$ for $f \in L^+$.

Now take any $f \in L^+(v, R)$. From the definition of the space $L(v, R)$ there exist a sequence $s_n \in S^+(V, R)$, a constant $M > 0$, and a v-null set A such that $s_n(x)$ converges to $f(x)$ if $x \in X \setminus A$ and $\|s_n\|_v \leqslant M$ for all n.

Since the integral $\int$ is complete, we have by Lemma 3 that $A \in V$ and $v(A) = 0$. This implies that $c_A \in S(V, R)$ and

$$h_n = (1 - c_A)\, s_n \in S^+(V, R)\,.$$

By Lemma 1, $h_n \in L$ and

$$\int h_n = \int h_n \, \mathrm{d}v \leqslant \int s_n \, \mathrm{d}v \leqslant M \qquad \text{for } n = 1, 2, \ldots .$$

Set $g_{mn} = h_m \cap \ldots \cap h_n$ for $n \geqslant m$. Since L is a lattice $g_{mn} \in L$, and

$$\int g_{mn} \leqslant \int h_m \leqslant M \qquad \text{for } m = 1, 2, \ldots .$$

The sequence of functions g_{mn} converges decreasingly, as n goes to infinity, to the function g_m, where

$$g_m(x) = \{\inf h_n(x) \colon n \geqslant m\} \qquad \text{for all } x \in X\,.$$

Since $0 \leqslant \int g_{mn} \leqslant M$, by the countable additivity of the integral, $g_m \in L$ and $\int g_m \leqslant M$ for $m = 1, 2, \ldots$.

Let $g = (1 - c_A) f$. Since h_n converges to g, it follows that g_m converges increasingly to g, so that $g \in L$. Since $c_A f(x) = 0$ $\int$-almost everywhere, by Theorem 3, $c_A f \in L$ and therefore $f \in L$. This proves the theorem.

In the following theorem, for a given measure μ we shall denote by $M(\mu, r)$ the space of real valued measurable functions and by $L(\mu, R)$ the space of real valued summable functions generated by μ. $\int f \mathrm{d}\mu$ will denote the integral of f with respect to μ.

Theorem 5. *Let $\int$ be a Lebesgue integral and v the volume generated by $\int$. Let μ be the measure with smallest domain extending the volume v. Then,*

$$D(\textstyle\int) = L(\mu, R) = L(v, R) \cap M(\mu, R) \quad \textit{and} \quad \int f = \int f \, \mathrm{d}\mu = f \, \mathrm{d}v \ \text{ for all } f \in D(\textstyle\int)\,.$$

Proof. Let v be defined on the ring V of summable sets generated by $\int$. It follows from the results of [3] that the measure with smallest domain extending v is defined on the sigma-ring

$$M = \{A \colon A = \bigcup_{n=1}^{\infty} A_n,\ A_n \in V\}$$

by the formula

$$\mu(A) = \sup\{v(B):\ B \subset A,\ B \in V\} \qquad \text{for } A \in M$$

and that the finite part of μ is v, that is the family of sets $\{A \in M: \mu(A) < \infty\}$ is V, and $v(A) = \mu(A)$ for all $A \in V$.

By theorem 2 of [4], $L(\mu, R) = L(v, R) \cap M(\mu, R)$ and $\int f = \int f \mathrm{d}v$ for all $f \in L(\mu, R)$. In order to prove the theorem we only have to show that

$$D^+(\int) = L^+(v, R) \cap M^+(\mu, R)$$

and

$$\int f = \int f \mathrm{d}v \qquad \text{for all} \qquad f \in D^+(\int).$$

Take any $f \in D^+(\int)$. The proof for showing that $f \in L^+(v, R)$ and $\int f = \int f \mathrm{d}v$ is exactly the same as in Theorem 4. To prove that $f \in M^+(\mu, R)$ we have to show that for any $a > 0$ the set

$$A_a = \{x \in X:\ f(x) > a\} \in M.$$

By theorem 2, $A_a \in V$. Since $V \subset M$, we have $A_a \in M$. Thus

$$D^+(\int) \subset L^+(v, R) \cap M^+(\mu, R)$$

and

$$\int f = \int f \mathrm{d}v \qquad \text{for} \qquad f \in D^+(\int).$$

Now take any $f \in L^+(v, R) \cap M^+(\mu, R)$. Since $f \in M^+(\mu, R)$, by lemma 1 and lemma 2 of [4], there exists a sequence $s_n \in S^+(V, R)$ increasingly convergent everywhere to the function f and such that $\int s_n \mathrm{d}v \leqslant \int f \mathrm{d}v$. By lemma 1, $s_n \in D^+(\int)$ and

$$\int s_n = \int s_n \mathrm{d}v \leqslant \int f \mathrm{d}v,$$

which implies by the countable additivity of the integral that $f \in D^+(\int)$. This completes the proof of the theorem.

In a subsequent paper we shall extend a non-complete integral to a complete one and find a representation for the completion.

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