Riv. Mat. Univ. Parma (2) 12 (1971), 137-144

R. S. L. SRIVASTAVA and S. K. BAJPAI (*)

The Asymptotic Values of an Entire Function Represented by Dirichlet Series. (**)

1. - Let

$$f(s)=\sum_{0}^{\infty} a_n \exp(s\lambda_n) \qquad (\lambda_0 \geqslant 0,\ \lambda_{n+1} > \lambda_n,\ \lim_{n\to\infty}\lambda_n=\infty)$$

be an entire function in the sense that the DIRICHLET series representing $f(s)$ converges absolutely for all finite s. Set

$$M(\sigma, f)=\underset{-\infty<t<\infty}{\text{l.u.b.}}\,|f(\sigma+it)|\,, \qquad \mu(\sigma, f)=\max_{n\geqslant 0}\left\{|a_n|\ \exp(\sigma\lambda_n)\right\},$$

$$\mu(\sigma+it)=\mu(s)=\mu(\sigma)\exp\{i\lambda_{\nu(\sigma)}t\}\,, \quad \nu(\sigma, f)=\max\{n\,|\mu(\sigma, f)\}=|a_n|\exp(\sigma\lambda_n)\,.$$

Clearly $\mu(s, f)$ is continuous in each of the strips where $\nu(\sigma, f)$ is continuous but in general it is discontinuous at the points where $\nu(\sigma, f)$ is. Let $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots \to \infty$ be the points where $\nu(\sigma, f)$ changes its value and the range of $\nu(\sigma, f)$ be $\{n_k\}$. We write $\sigma_n=\sigma(n)$ and define the following symbols:

$$\begin{matrix} L \\ l \end{matrix} = \lim_{k\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \left(\sigma(n_{k+1})-\sigma(n_k)\right),$$

$$\begin{matrix} A \\ a \end{matrix} = \lim_{k\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\lambda_{n_{k+1}}-\lambda_{n_k}}{\lambda_{n_k}-\lambda_{n_{k-1}}}, \qquad \begin{matrix} B \\ b \end{matrix} = \lim_{k\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} (\lambda_{n_{k+1}}-\lambda_{n_k}),$$

$$\begin{matrix} E \\ c \end{matrix} = \lim_{k\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \left\{(\lambda_{n_{k+1}}-\lambda_{n_k})\left(\sigma(n_{k+1})-\sigma(n_k)\right)\right\}.$$

(*) Indirizzo degli AA.: Department of Mathematics, Indian Institute of Technology, Kanpur, India.

(**) Ricevuto: 7-VI-1969.

Throughout this paper we shall assume $b > 0$ and define the asymptotic values of $f(s)$ as follows:

Let $r(p)$ be any continuous path in the complex plane such that as $p \to \infty$, $r(p) \to \infty$. Thus, if $f(r(p))/\mu(r(p))$ tends to a limit ω as $p \to \infty$, we say that ω is a μ-asymptotic value of $f(s)$ and $r(p)$ the corresponding μ-asymptotic path. We shall prove the following theorems:

Theorem 1. *If $L > 0$ and $B < \infty$, then $f(s)$ has no μ-asymptotic value.*

Theorem 2. *If $0 < E = c < \infty$ and $1 \leq a = A < \infty$ and $f(s)$ has the form*

$$f(s) = 1 + \sum_{k=1}^{\infty} \exp\{s\,\lambda_{n_k} - \sigma(1)\,\lambda_1 - \sigma(2)\,(\lambda_2 - \lambda_1) - \ldots - \sigma(n_k)\,(\lambda_{n_k} - \lambda_{n_{k-1}})\}\,,$$

then $f(s)$ has no μ-asymptotic value.

2. – In proving the above theorems we need the following

Lemma. *If $0 \leq \sigma < \{\sigma(n_{k+1}) - \sigma(n_k)\}$, then*

$$\frac{\mu(\sigma + \sigma(n_k), f)}{M(\sigma + \sigma(n_k), f)} \leq \frac{\pi}{4}\,[1 + \exp(\sigma\{\lambda_{n_{k+1}} - \lambda_{n_k}\})]\,. \tag{2.1}$$

Proof. Let $\mu(\sigma(n_k), f) = |a_{n_{k-1}}|\exp\{\sigma(n_k)\,\lambda_{n_{k-1}}\} = |a_{n_k}|\exp\{\sigma(n_k)\,\lambda_{n_k}\}$.

Then [1]

$$(2.2)\begin{cases} |a_{n_k}\exp\{(\sigma + \sigma(n_k) + im_1)\,\lambda_{n_k}\} + a_{n_{k-1}}\exp\{(\sigma + \sigma(n_k) + im_2)\,\lambda_{n_{k-1}}\}| \\[2ex] = \left|\lim\limits_{T\to\infty}\dfrac{1}{2T}\displaystyle\int_{-T}^{T} f(\sigma+\sigma(n_k)+it)\,\{\exp\{-\lambda_{n_k}(t+m_1)\,i\} + \exp\{-\lambda_{n_{k-1}}(t+m_2)\,i\}\}\,\mathrm{dt}\right| \\[2ex] \leq 2M(\sigma + \sigma(n_k), f)\lim\limits_{T\to\infty}\left[\dfrac{1}{2T}\displaystyle\int_{-T}^{T}\left|\cos\dfrac{1}{2}\,\{(\lambda_{n_k} - \lambda_{n_{k-1}})\,(t + m_3)\}\right|dt\right] \\[2ex] = \dfrac{4}{\pi}\,M(\sigma + \sigma(n_k), f)\,. \end{cases}$$

Also

$$|a_{n_k}\exp\{(\sigma + \sigma(n_k) + im_1)\,\lambda_{n_k}\} + a_{n_{k-1}}\exp\{(\sigma + \sigma(n_k) + im_2)\,\lambda_{n_{k-1}}\}| =$$

$$= \mu(\sigma + \sigma(n_k), f)\left|1 + \frac{a_{n_{k-1}}}{a_{n_k}}\,\frac{\exp\{\sigma(n_k)\,\lambda_{n_{k-1}}\}}{\exp\{\sigma(n_k)\,\lambda_{n_k}\}}\,\frac{\exp\{(\sigma + im_2)\,\lambda_{n_{k-1}}\}}{\exp\{(\sigma + im_1)\,\lambda_{n_k}\}}\right|.$$

Where m_1, m_2 are arbitrary real numbers, choosing in such a manner that $m_2\lambda_{n_{k-1}} - m_1\lambda_{n_k} = \arg a_{n_k} - \arg a_{n_{k-1}}$, then the right hand side of the above expression becomes,

$$\mu(\sigma + \sigma(n_k), f)\left[1 + \exp\{\sigma(\lambda_{n_{k-1}} - \lambda_{n_k})\}\right]. \tag{2.3}$$

On combining (2.2) and (2.3), the Lemma follows.

Proof of Theorem 1. Without loss of generality we may assume that $f(-\infty) = 1$. Let $0 < \alpha < \beta < L_1 < L$ and $\alpha < (1/B)\log\{\pi/(4-\pi)\}$. Under the hypothesis of the Theorem, there exists a sequence of integers $\{k_m\}$ such that $\sigma(n_{k_m+1}) - \sigma(n_{k_m}) > L_1$. Contrary to the Theorem suppose that $f(s)$ has a μ-asymptotic value ω and so there exists a path $r(p)$ corresponding to ω with $r(p) \to \infty$ and $\{f(r(p))/\mu(r(p))\} \to \omega$. Denoting

$$\Phi_m(G) = \frac{f(\sigma(n_{k_m}) + G)}{\mu(\sigma(n_{k_m}) + G)} \qquad \text{for } G \in \Omega_1,$$

$$\Omega_1 = \{G \mid 0 < G_r < L_1,\ \operatorname{Re} G = G_r\},$$

we have [2] $(k = n_{k_m})$

$$|f(\sigma(k) + G)| \leqslant$$
$$\leqslant 1 + \sum_{i=1}^{\infty} \exp\{\lambda_i(\sigma(k) + G_r) - \sigma(1)\lambda_1 - \sigma(2)(\lambda_2 - \lambda_1) - \ldots - \sigma(i)(\lambda_i - \lambda_{i-1})\},$$

$$|\mu(\sigma(k) + G)| =$$
$$= \exp\{\lambda_k(\sigma(k) + G_r) - \sigma(1)\lambda_1 - \sigma(2)(\lambda_2 - \lambda_1) - \ldots - \sigma(k)(\lambda_k - \lambda_{k-1})\}.$$

Hence:

$$|\Phi_m(G)| \leqslant \exp\{\lambda_1\sigma(1) + (\lambda_2 - \lambda_1)\sigma(2) + \ldots + (\lambda_k - \lambda_{k-1})\sigma(k) - \lambda_k(\sigma(k) + G_r)\} +$$
$$+ \sum_{j=1}^{\infty} \exp\{\lambda_j(\sigma(k) + G_r) + \lambda_1\sigma(1) + (\lambda_2 - \lambda_1)\sigma(2) + \ldots + (\lambda_k - \lambda_{k-1})\sigma(k) -$$
$$- \lambda_k(\sigma(k) + G_r) - \lambda_1\sigma(1) - (\lambda_2 - \lambda_1)\sigma(2) - \ldots - (\lambda_j - \lambda_{j-1})\sigma(j)\}$$

$$\leqslant 1 + \sum_{j=1}^{\infty} \exp\{(G_r - L)(\lambda_{j+k} - \lambda_k)\} + \sum_{j=1}^{\infty} \exp\{G_r(\lambda_k - \lambda_{k+j})\} \equiv C'(G_r).$$

The expression of $C'(G_r)$ shows that for any k and G_r it converges uniformly because of $L > 0$ and $b > 0$. Hence a least upper bound $C(G_r)$ of $C'(G_r)$ can be found and $\Phi_m(G)$ is uniformly bounded on a closed and bounded subset of Ω_1. Thus there is a subsequence $\{\Phi_m(G)\}$ which converges uniformly on a closed and bounded subset of Ω_1 to a function $H(G)$ represented by DIRICHLET series. We shall show that $H(G)$ is non-constant. Suppose $H(G) = p$ (constant). For $0 < \sigma < L_1$, we have

$$p = \lim_{T\to\infty} \left[\frac{1}{2T}\int_{-T}^{T} H(G)\,dt\right] = \lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T}\{\lim_{m\to\infty}\Phi_m(G)\}\,dt\right]$$

$$= \lim_{m\to\infty}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Phi_m(G)\,dt\right] = 1 .$$

Because the only constant term in the expansion of $\Phi_m(G)$ is 1. Thus, $H(G) \equiv 1$ on Ω_1. But by (2.1)

$$M(\sigma, H) = \lim_{m\to\infty} M(\sigma, \Phi_m(G)) \geqslant \frac{\pi}{4}(1 - e^{B\sigma}) \qquad \text{for } 0 < \sigma < L_1 .$$

In particular, if $\sigma < (1/B)\log\{\pi/(4-\pi)\} = \alpha$. Then

$$M(\sigma, H) > \frac{\pi}{4}\left(1 + \frac{4-\pi}{\pi}\right) = 1 .$$

Hence $H(G)$ must be non-constant.

Let $\Omega = \{G \mid \alpha \leqslant G_r \leqslant \beta\}$ and since $f(s)$ has a μ-asymptotic value there exists an unbounded set I with the following property:

For each $p \in I$, there is a unique integer k_m such that $\sigma(n_{k_m}) \leqslant \log|\gamma(p)| < \sigma(n_{k_m+1})$. Writing $\sigma(n_{k_m}) + \log\gamma_m(p) = \log\gamma(p)$. We have for large m, $0 \leqslant \log|\gamma_m(p)| \leqslant L + o(1)$. Hence $\gamma_m(p)$ is bounded. We now consider the set X of limit points of $\{\gamma_m(p)\}$ as $p \to \infty$, $p \in I$, and which lie in Ω and prove that the set X of limit points is uncountable and on this set $H(G)$ is constant. To prove this, let Y be the intersection of real axis and the set Ω and define mapping F; $F: Y \to X$ such that for each $y \in Y$, there exists a unique integer m, for which $p_m \in I$ and $\log|\gamma_{p_m}| = y + \sigma(n_{k_m})$, then $\log|\gamma_m(p_m)| = y$. Choose a limit point ϑ of $\{\gamma_m(p_m)\}$ and define $F(y) = \vartheta$. Then F is one-one, since $\log|F(y)| = y$. Thus X is uncountable since Y is. Furthermore, $H(G)$

is constant. For suppose $\gamma_m(p_s) \to p \in X$ for a sequence $\{p_s\}$ with $p_s \in I$. By virtue of uniform convergence, $\Phi_m(\gamma_m(p_s)) \to H(b)$. But we are assuming ω to be a μ-asymptotic value and so $H(b) = \omega$. Hence H is constant on X. This is a contradiction. Hence $f(s)$ has no μ-asymptotic value.

3. - Proof of Theorem 2.

Suppose $f(s)$ has μ-asymptotic value ω $(0 \leqslant |\omega| < \infty)$. Let $\gamma(p)$ be the corresponding μ-asymptotic path to ω. For given p, take m to be the unique integer for which $\sigma(n_m) \leqslant \log|\gamma(p)| < \sigma(n_{m+1})$, and define

$$\log \gamma_m(p) = x + iD\,, \qquad \text{where} \qquad \log\gamma(p) = \sigma(n_m) + x\,c\,(\lambda_{n_m+1} - \lambda_{n_m})^{-1} + iD\,,$$

$-\infty < D < \infty$. Then it follows that $0 \leqslant x < 1 + o(1)$. Hence $\log|\gamma_m(p)|$ is bounded above and $\gamma_m(p)$ is bounded for all m and p. Now writing $P_m(G) = f(s)/\mu(s)$, where $s = \sigma(n_m) + cG\,(\lambda_{n_m+1} - \lambda_{n_m})^{-1}$. We get $0 \leqslant cG_r\,(\lambda_{n_m+1} - \lambda_{n_m})^{-1}$. Therefore, for sufficiently large value of m, $0 \leqslant cG_r\,(\lambda_{n_m+1} - \lambda_{n_m})^{-1} < \sigma(n_{m+1}) - \sigma(n_m)$ or $\sigma(n_m) \leqslant \operatorname{Re} s < \sigma(n_{m+1})$. Hence, for $\nu(\operatorname{Re} s) = n_m$,

$$\mu(s, f) = \exp\{s\,\lambda_{n_m} - \sigma(1)\,\lambda_1 - \sigma(2)\,(\lambda_2 - \lambda_1) - \ldots - \sigma(n_k)\,(\lambda_{n_m} - \lambda_{n_{m-1}})\}\,.$$

Write

$$a^j(m) = \begin{cases} (\lambda_{n_{m+1}} - \lambda_{n_m})\,\sigma(n_{m+1}) + \ldots + (\lambda_{n_{m+j}} - \lambda_{n_{m+j-1}})\,\sigma(n_{m+j}) & (j > 0) \\ 0 & (j = 0) \\ (\lambda_{n_{m-1}} - \lambda_{n_m})\,\sigma(n_m) + \ldots + (\lambda_{n_{m+j}} - \lambda_{n_{m+j+1}})\,\sigma(n_{m+j+1}) & (j < 0)\,. \end{cases}$$

Then

$$\text{(3.1)} \qquad \frac{f(s)}{\mu(s)} = \sum_{-(m-1)}^{\infty} \exp\left\{(\lambda_{n_{m+j}} - \lambda_{n_m})\,\sigma(n_m) + \frac{\lambda_{n_{m+j}} - \lambda_{n_m}}{\lambda_{n_{m+1}} - \lambda_{n_m}}\,cG - a^j(m)\right\}.$$

Since $0 < E = c < \infty$ and $1 \leqslant a = A < \infty$, there exist numbers A_1, A_2, q such that

$$0 < A_1 < \frac{\lambda_{n_{m+1}} - \lambda_{n_m}}{\lambda_{n_m} - \lambda_{n_{m-1}}} < A_2 < \infty \qquad \text{and} \qquad 0 < q < (\lambda_{n_{m+1}} - \lambda_{n_m})\,(\sigma(n_{m+1}) - \sigma(n_m))$$

for $m=1, 2, 3, \ldots$: Let $j \geqslant 2$. Then

$$(\lambda_{n_{m+j}} - \lambda_{n_m})\sigma(n_m) - a^j(m) \leqslant -\sum_{i=2}^{j} (\lambda_{n_{m+i}} - \lambda_{n_{m+i-1}})(\sigma(n_{m+i}) - \sigma(n_{m+i-1})) .$$

$$\leqslant -(j-1)q .$$

Similarly we have, for $j \leqslant -2$,

$$(\lambda_{n_{m+j}} - \lambda_{n_m})\sigma(n_m) - a^j(m) \leqslant \sum_{i=-j}^{1} (\lambda_{n_{m-i}} - \lambda_{n_{m-i+1}})(\sigma(n_{m-i+1}) - \sigma(n_{m-i}))$$

$$\leqslant \frac{(-j+1)q}{A_2} .$$

Hence we have

$$(3.2) \qquad |\exp\{\sigma(n_m)(\lambda_{n_{m+j}} - \lambda_{n_m}) - a^j(m)\}| \leqslant \begin{cases} \exp\{-q(j-1)\} & (j \geqslant 2) \\ 1 & (j=0) \\ \exp\{q(j+1)/A_2\} & (j \leqslant -2) . \end{cases}$$

Therefore by WEIERSTRASS M-test (3.1) converges uniformly both in m and G, and we get

$$\lim_{m\to\infty} \frac{f(s)}{\mu(s)} = \sum_{-\infty}^{\infty} \exp\left\{\sigma(n_m)(\lambda_{n_{m+i}} - \lambda_{n_m}) + \frac{\lambda_{n_{m+j}} - \lambda_{n_m}}{\lambda_{n_{m+1}} - \lambda_{n_m}} cG - a^j(m)\right\} .$$

Further, for $j > 0$,

$$(\lambda_{n_{m+j}} - \lambda_{n_m})\sigma(n_m) - a^j(m) =$$

$$= -\sum_{i=1}^{j} \sum_{p=i}^{q} \left[\prod_{s=1}^{p-1} \frac{\lambda_{n_{m+s+1}} - \lambda_{n_{m+s}}}{\lambda_{n_{m+s}} - \lambda_{n_{m+s-1}}} \left\{(\sigma(n_{m+q}) - \sigma(n_{m+q-1}))(\lambda_{n_{m+q}} - \lambda_{n_{m+q-1}})\right\}\right]$$

and so

$$(3.3) \qquad \lim_{m\to\infty} \{(\lambda_{n_{m+j}} - \lambda_{n_m})\sigma(n_m) - a^j(m)\} =$$

$$= \begin{cases} -\dfrac{1}{2} j(j+1)c & \text{if } a = A = 1, \\ -\dfrac{A^{j+1} - A(j+1) + j}{(A-1)^2} c & \text{if } 1 < a = A < \infty . \end{cases}$$

A similar argument shows that (3.3) is also valid when $j < 0$. Hence we have

$$(3.4)\qquad \lim_{m\to\infty} \frac{f(s)}{\mu(s)} =$$

$$= \begin{cases} \sum_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\, j(j+1)\, c + c j q\right] & (a = A = 1) \\ \sum_{-\infty}^{\infty} \exp\left[-c\, \frac{A^{j+1} - A(j+1) + j}{(A-1)^2} + \frac{A(A^j - 1)}{A-1}\, cG\right] & (1 < a = A < \infty) \end{cases}$$

$$\equiv Q(s)\,.$$

It is easy to see that $Q(s)$ is non-constant and that the set T of limit points of $\{\gamma_m(p)\}$ is uncountable, and $Q(s)$ is constant on T. Thus contradicting the fact that $Q(s)$ is non-constant on Ω. Hence $f(s)$ has no μ-asymptotic value. The argument of the proof runs exactly in the same way as in the Theorem 1.

4. - Examples.

The following examples illustrate that the class of entire DIRICHLET series for which μ-asymptotic values do not exist, is non-vacous and at the same time exist DIRICHLET series for which μ-asymptotic value exists.

Example 1. Let

$$f(s) = \sum_{n=0}^{\infty} \lambda^{-\frac{1}{2} n(n+1)} \exp(ns) \qquad (1 < \lambda < \infty)\,.$$

Clearly $\sigma(n) = n \log \lambda$ and $\lim_{n\to\infty} \exp\{n \log\lambda - (n-1) \log \lambda\} = \lambda > 1$, implying $L > 0$ and $\liminf(\lambda_{n+1} - \lambda_n) = 1$ and so $b > 0$ and $B < \infty$. Hence all the conditions of Theorem 1 are satisfied and so $f(s)$ has no μ-asymptotic value.

Example 2. Let

$$f(s) = \sum_{n=0}^{\infty} \exp(n^2 s)\, n^{-2\alpha n^2} \qquad (0 < \alpha < \infty)\,.$$

In this case $(\log|a_n/a_{n+1}|)/(\lambda_{n+1} - \lambda_n)$ is non-decreasing and so $\sigma(n) = 2\alpha(n^2 \log n - (n-1)^2 \log(n-1))/(2n-1)$, $a = A = 1$ and $E = e = 4\alpha$. Thus all the conditions of Theorem 2 are satisfied (together with $b > 0$). Hence $f(s)$ has no μ-asymptotic value.

Example 3. If k is any complex number, then considering the function

$$f(s) = \sum_{n=0}^{\infty} \left[\frac{\exp\,[2ns + (2n+1)(\pi/2)i]}{n!} + \frac{k}{\sqrt{2\pi}} \left\{ \frac{\exp\{(2n-1)s\}}{n!} - \frac{\exp(-s)}{1} \right\} \right]$$

and, choosing $\sigma > \log(|k|/\sqrt{2\pi})$, we have

$$\mu(\sigma, f) \sim \sum_{n=0}^{\infty} \frac{\exp(2n\sigma - \sigma)}{n!\sqrt{2\pi}}.$$

Clearly it follows that $\lim\limits_{\sigma \to \infty} \{f(\sigma)/\mu(\sigma)\} \to k$. Hence k is a μ-asymptotic value.

References.

[1] G. SANSONE and J. GERRETSEN, *Lectures on the Theory of Functions of a Complex Variable*, P. Noordhoff, Groningen 1960.

[2] A. G. AZPEITIA, *On the maximum modulus and the maximum term of an entire Dirichlet series*, Proc. Amer. Math. Soc. **12** (1961), 717-721.

* * *