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Quasi-Hereditary Rings. (**)

In this paper we shall discuss some characterizations of a quasi-hereditary ring and then we shall find out some relations between quasi-hereditary and other types of rings.

Throughout the paper R is a ring with unity $1 \neq 0$ and all modules are left unital R-modules unless otherwise stated.

A module P is said to be *finitely related* if there exists an exact sequence of R-modules

$$0 \to K \to F \to P \to 0$$

with F free and finitely generated and K finitely generated.

It can be shown that if P is finitely related and if there exists an exact sequence

$$0 \to K \to F \to P \to 0$$

with F free and finitely generated then K must be finitely generated [1].

A module M is said to be *flat* it $\mathrm{Tor}_1^R(A, M) = 0$ for all right R-modules A.

The following lemma is due to CHASE [4].

Lemma 1. *A finitely related flat module is projective.*

An immediate consequence of this lemma is the following

Lemma 2. *A finitely generated module is projective if and only if it is finitely related and flat.*

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Proof. In the light of Lemma 1, it is sufficient to prove the the converse part.

Let A be a finitely generated projective module: then A is flat [6].

The exact sequence

$$0 \to K \to F \to A \to 0$$

where F is free and finitely generated splits as A is projective.

Splitting of the sequence implies that K is a direct summand of F. So K is finitely generated, hence A is finitely related.

Definition. A ring R in which every left ideal is flat as R-module is called a *quasi-hereditary ring.* Actually we should have called it left quasi-hereditary but soon we shall see that there is no distinction between a left and a right quasi-hereditary ring, so the reason for nomenclature.

Theorem 1. *Following conditions are equivalent for a ring R:*

(a) *Every left ideal is flat.*

(b) *Every finitely related left ideal is projective.*

(c) $GWD(R) \leqslant 1$.

(d) *Every module which can be embedded in a direct product of finite number of copies of ${}_RR$ is flat.*

(e) *Every submodule of a flat module is flat.*

(f) *Every right ideal is flat.*

(g) *Every finitely related right ideal is projective.*

(h) *Every right R-module which can be embedded in a direct product of finite number of copies of R_R is flat.*

(i) *Every submodule of a flat right module is flat.*

Proof. We shall prove the Theorem in the following order:

(a) ⇔ (b) ; (a) ⇔ (c) ; (c) ⇒ (e), (e) ⇒ (a), (e) ⇒ (d) ; (d) ⇒ (a)

and then the rest will follow because of left and right symmetry of condition (c).

(a) ⇒ (b) follows from Lemma 1.

(b) ⇒ (a): every left ideal can be regarded as direct limit of finitely related left sub-ideals [2], so (b) implies that each left ideal is direct limit of flat left sub-ideals. Since the functor Tor commutes with direct limit we get (a).

(a) $\Rightarrow$ (c). Let M be a cyclic module, then $M \approx R/I$ for some left ideal I of R.

Consider the exact sequence

$$0 \to I \to R \to R/I \to 0 .$$

This gives rise to the exact sequence

$$\ldots \to \mathrm{Tor}_2(A, R) \to \mathrm{Tor}_2(A, R/I) \to \mathrm{Tor}_1(A, I) \to \ldots ,$$

where A is any right R-module. The flatness of R and $I \Rightarrow \mathrm{Tor}_2(A, R/I) = 0$.

Thus $\mathrm{Tor}_2(A, M) = 0$ for all right R-modules A.

Next let M be a finitely generated module with generators $m_1, m_2, \ldots, m_n$. Let N be the submodule of M generated by $m_1, m_2, m_2, \ldots, m_{n-1}$.

Consider the exact sequence

$$0 \to N \to M \to M/N \to 0 .$$

Since M/N is cyclic, $\mathrm{Tor}_2(A, M/N) = 0$ for all right R-modules A. The above exact sequence gives rise to the exact sequence

$$\ldots \to \mathrm{Tor}_2(A, N) \to \mathrm{Tor}_2(A, M) \to \mathrm{Tor}_2(A, M/N) \to \ldots . \tag{1}$$

Now if we assume that for all modules K generated by $n-1$ elements, $\mathrm{Tor}_2(A, K) = 0$ for all right R-modules A. Then in particular $\mathrm{Tor}_2(A, N) = 0$. So by (1) $\mathrm{Tor}_2(A, M) = 0$. Hence by induction, for every finitely generated module M, $\mathrm{Tor}_2(A, M) = 0$ for all right R-modules A.

Finally as every module is direct limit of finitely generated submodules and the functer Tor commutes with direct limit, $\mathrm{Tor}_2(A, M) = 0$ for all modules M. So $GWD(R) \leqslant 1$.

(c) $\Rightarrow$ (a). Let I be a left ideal of R. The exact sequence

$$0 \to I \to R \to R/I \to 0$$

induces the exact sequence

$$\ldots \to \mathrm{Tor}_2(A, R/I) \to \mathrm{Tor}_1(A, I) \to \mathrm{Tor}_1(A, R) \to \ldots ,$$

where A is any right R-module.

Now (c) implies $\mathrm{Tor}_2(A, R/I) = 0$ and flatness of R implies that $\mathrm{Tor}_1(A, R) = 0$. So $\mathrm{Tor}_1(A, I) = 0$ for all right R-modules A. Hence I is flat.

(c) $\Rightarrow$ (e). Let M be a flat module and let N be its submodule.

The exact sequence $0 \to N \to M \to M/N \to 0$ induces the exact sequence

$$\dots \to \mathrm{Tor}_2(A, M/N) \to \mathrm{Tor}_1(A, N) \to \mathrm{Tor}_1(A, M) \to \dots,$$

where A is any right R-module.

Since M is flat, $\mathrm{Tor}_1(A, M) = 0$.

Also (c) implies $\mathrm{Tor}_2(A, M/N) = 0$.

So $\mathrm{Tor}_1(A, N) = 0$ for all right R-modules A. Hence N is flat.

(e) $\Rightarrow$ (a). Trivial as every left ideal of R is submodule of ${}_RR$.

(e) $\Rightarrow$ (d). Let M be embeddable in a direct product of finite number of copies of ${}_RR$. Now as ${}_RR$ is flat, every direct product of finite number of copies of ${}_RR$ is flat. So M is flat by (e).

(d) $\Rightarrow$ (a). Clearly every left ideal satisfies condition (d).

Corollary 1. *A left (right) semi-hereditary ring is quasi-hereditary.*

Proof. Each finitely generated left (right) ideal is projective hence flat. Now as the functor Tor commutes with direct limits and each left (right) ideal is direct limit of finitely generated left (right) sub-ideals, corollary follows by condition (a) of the Theorem.

Definition. A ring R is said to be *rpp*, *rpfr*-ring if every right principal ideal is projective, finitely related respectively.

Corollary 2. *A quasi-hereditary rpfr-ring is rpp-ring.*

This follows from condition (b) of Theorem 1.

A ring R is said to be *left perfect* if each module M has weak dimensions and projective dimension to be equal [3]. It is proved by Bass [3] that for a left perfect ring R every flat module is projective.

Corollary 3. *A quasi-hereditary left perfect ring is left-hereditary.*

The corollary follows from condition (a) of Theorem 1.

Theorem 2. *A left Noetherian quasi-hereditary ring is left-hereditary.*

Proof. Let I be a left ideal of R. Consider the exact sequence

$$0 \to K \to F \to I \to 0,$$

where F is free and finitely generated module as I is finitely generated.

Now as R is left Noetherian every sub-module of a finitely generated module is finitely generated [5]. Hence I is finitely related and so by Theorem 1, I is projective.

Theorem 3. *A quasi-hereditary commutative integral domain is a Prufer domain.*

Proof. It is proved by CHASE [4] that a commutative integral domain R is a Prufer domain if and only if $GWD(R) \leqslant 1$. Consequently the Theorem follows by condition (c) of Theorem 1.

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