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Generalized Logarithmic Mean Function of Derivatives of Entire Functions Defined by Dirichlet Series. (**)

1. - Let E be the set of mappings f: $C \to C$ (C is the complex field) such that the image under f of an element $s \in C$ is

$$f(s) = \sum_{n \in N} a_n \exp(s\,\lambda_n)$$

with

(1.1)
$$\limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D < +\infty,$$

and $\sigma_c^f = +\infty$ (σ_c^f is the abscissa of convergence of the DIRICHLET series defining f); N is the set of natural numbers $0, 1, 2, \ldots$; $\langle \lambda_n :\ n \in N \rangle$ is a strictly increasing unbounded sequence of nonnegative reals; $s = \sigma + it$, where σ, $t \in R$ (R is the field of reals); and $\langle a_n :\ n \in N \rangle$ is a sequence in C. Since the DIRICHLET series defining f converges for each complex s, f is an entire function. Also, since $D < +\infty$, we have ([1], p. 168), $\sigma_a^f = +\infty$ (σ_a^f is the abscissa of absolute convergence of the DIRICHLET series defining f), and that f is bounded on each vertical line $\sigma = \sigma_0$.

In an earlier paper [2] we have defined the generalized logarithmic mean function of an entire function $f \in E$ and have investigated into some of its properties. In this paper we define analogously the generalized logarithmic mean function of the n^{th} derivative $f^{(n)}$ of an entire function $f \in E$ and study some of its properties.

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Definition. For any $f \in E$, $\sigma < \sigma_c^f$ and $\delta \in R_+$ (R_+ is the set of positive reals), we define the generalized logarithmic mean function G_n of the n^{th} derivative $f^{(n)}$ of f as

$$(1.2)\quad G_n(\sigma, f^{(n)}) = \lim_{T\to+\infty} \frac{1}{2T\exp(\delta\sigma)} \int_0^\sigma \int_{-T}^{T} \log |f^{(n)}(x+it)| \exp(\delta x)\, dx\, dt\,, \quad \forall\, n \in N\,.$$

Our study of a few properties of G_n has taken the shape of the following theorems:

Theorem 1. *If G_n is the generalized logarithmic mean function of the n^{th} derivative $f^{(n)}$ of an entire fuction $f \in E$, then G_n is an increasing function and* $\log G_n$ *is a convex function of σ.*

The proof of this Theorem is similar to that of theorem 1 of [2].

Theorem 2. *If G_n is the generalized logarithmic mean function of the n^{th} derivative $f^{(n)}$ of an entire function $f \in E$, and the lower order λ of f is greater than* 1, *then*

$$(1.3)\qquad G_n(\sigma, f^{(n)}) > G_{n-1}(\sigma, f^{(n-1)}) > \ldots > G(\sigma, f)\,, \qquad \forall\ \sigma \geqslant \sigma_0 > 0\,,$$

where $G_0 = G$.

In order to prove this Theorem we need the following two lemmas:

Lemma 1. *If G_1 is the generalized logarithmic mean function of the derivative $f^{(1)}$ of an entire function $f \in E$, then*

$$(1.4)\qquad G_1(\sigma, f^{(1)}) \geqslant \frac{G(\sigma, f)\log G(\sigma, f)}{\sigma}\,, \qquad \forall\ \sigma \text{ s.t. } 0 < \sigma < \sigma_c^f\,.$$

Proof. We have, from (1.2),

$$G_1(\sigma, f^{(1)}) = \lim_{T\to+\infty} \frac{1}{2T\exp(\delta\sigma)} \int_0^\sigma \int_{-T}^{T} \log |f^{(1)}(x+it)| \exp(\delta x)\, dx\, dt$$

$$= \lim_{T\to+\infty} \frac{1}{2T\exp(\delta\sigma)} \int_0^\sigma \int_{-T}^{T} \log \left| \lim_{\varepsilon\to 0} \frac{f(x+it) - f(x(1-\varepsilon)+it)}{\varepsilon x} \right| \exp(\delta x)\, dx\, dt$$

$$\geqslant \lim_{T\to+\infty} \frac{1}{2T\exp(\delta\sigma)} \int_0^\sigma \int_{-T}^{T} \log \lim_{\varepsilon\to 0} \frac{|f(x+it)| - |f(x(1-\varepsilon)+it)|}{\varepsilon x} \exp(\delta x)\, dx\, dt$$

$$= \lim_{T\to+\infty} \frac{1}{2T\exp(\delta\sigma)} \int_0^\sigma \int_{-T}^{T} \lim_{\varepsilon\to 0} \log \frac{|f(x+it)| - |f(x(1-\varepsilon)+it)|}{\varepsilon x} \exp(\delta x)\, dx\, dt$$

$$\geqslant \lim_{T\to+\infty} \frac{1}{2T\exp(\delta\sigma)} \int_0^\sigma \int_{-T}^{T} \lim_{\varepsilon\to 0} \frac{\log|f(x+it)| - \log|(f(x(1-\varepsilon)+it)|}{\varepsilon x} \exp(\delta x)\, dx\, dt$$

$$= \lim_{\varepsilon\to 0} \lim_{T\to+\infty} \frac{1}{2T\cdot\exp(\delta\sigma)} \int_0^\sigma \int_{-T}^{T} \frac{\log|f(x+it)| - \log|f(x(1-\varepsilon)+it)|}{\varepsilon x} \exp(\delta x)\, dx\, dt$$

$$\geqslant \lim_{\varepsilon\to 0} \frac{G(\sigma, f) - G(\sigma(1-\varepsilon), f)}{\varepsilon\sigma}.$$

Let

$$\Phi(\sigma, f) = \frac{\log G(\sigma, f)}{\sigma},$$

then, since $\log G$ is an increasing convex function of σ ([**2**], theorem 1) it follows that φ is an increasing function of σ. Therefore

$$G_1(\sigma, f^{(1)}) \geqslant \lim_{\varepsilon\to 0} \frac{\exp(\sigma\varphi(\sigma)) - \exp((\sigma - \varepsilon\sigma)\varphi(\sigma))}{\varepsilon\sigma} =$$

$$= \exp(\sigma\varphi(\sigma))\, \varphi(\sigma) = \frac{G(\sigma, f)\log G(\sigma, f)}{\sigma}.$$

Lemma 2. *If G_1 is the generalized logarithmic mean function of the derivative $f^{(1)}$ of an entire function $f \in E$, and if ϱ and λ are, respectively, the Ritt order and the lower order of f, then*

$$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{G_1(\sigma, f^{(1)})}{G(\sigma, f)} \geqslant \begin{matrix}\varrho\\ \lambda\end{matrix}. \tag{1.5}$$

This follows from (1.4) and the following result in [**2**]:

$$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log G(\sigma, f)}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}. \tag{1.6}$$

Proof of Theorem 2. If $f^{(k)}$ is the k^{th} derivative of f, then, writing (1.5) for $f^{(k)}$, we get

$$\lim_{\sigma\to+\infty} \inf \frac{G_k(\sigma, f^{(k)})}{G_{k-1}(\sigma, f^{(k-1)})} \geqslant \lambda.$$

Hence, for any $\varepsilon > 0$, we have

$$G_k(\sigma, f^{(k)}) > (\lambda - \varepsilon)\, G_{k-1}(\sigma, f^{(k-1)})\,, \qquad \forall\ \sigma \geqslant \sigma_0(\varepsilon),$$

or, since ε is arbitrary and $\lambda > 1$,

$$G_k(\sigma, f^{(k)}) > G_{k-1}(\sigma, f^{(k-1)})\,. \tag{1.7}$$

Writing (1.7) for $k = 1, 2, \ldots, n$, we get n inequalities and combining these n inequalities we get (1.3).

Theorem 3. *If G_n is the generalized logarithmic mean function of the n^{th} derivative $f^{(n)}$ of an entire function $f \in E$, and the lower order λ of f is greater than* 1, *then*

$$G_n(\sigma, f^{(n)}) > G(\sigma, f)\left(\frac{\log G(\sigma, f)}{\sigma}\right)^n, \qquad \forall \sigma \geqslant \sigma_0 > 0\,. \tag{1.8}$$

Proof. If $f^{(k)}$ is the k^{th} derivative of f, then we get, from (1.4),

$$\frac{G_k(\sigma, f^{(k)})}{G_{k-1}(\sigma, f^{(k-1)})} \geqslant \frac{\log G_{k-1}(\sigma, f^{(k-1)})}{\sigma}\,. \tag{1.9}$$

Giving k the values $1, 2, \ldots, n$ in (1.9) and multiplying the n inequalities thus obtained we get

$$\frac{G_n(\sigma, f^{(n)})}{G(\sigma, f)} \geqslant \frac{\log G(\sigma, f)}{\sigma} \cdot \frac{\log G_1(\sigma, f^{(1)})}{\sigma} \cdots \frac{\log G_{n-1}(\sigma, f^{(n-1)})}{\sigma}$$
$$> \left(\frac{\log G(\sigma, f)}{\sigma}\right)^n,$$

in view of (1.3). Hence the Theorem.

Corollary. *If G_n is the generalized logarithmic mean function of the n^{th} derivative $f^{(n)}$ of an entire function $f \in E$, and the lower order λ of f is greater than* 1, *then*

$$\lim_{\sigma \to +\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(\frac{G_n(\sigma, f^{(n)})}{G(\sigma, f)}\right)^{1/n} \geqslant \begin{matrix}\varrho\\ \lambda\end{matrix}\,. \tag{1.10}$$

This follows from (1.8) and (1.6).

2. – In this section we shall study results pertaining to the logarithmic mean function L_n and the generalized logarithmic mean function G_n of the n^{th} derivative $f^{(n)}$ of an entire function $f \in E$. The function L_n is defined as follows:

$$L_n(\sigma, f^{(n)}) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \log |f^{(n)}(\sigma + it)| \, dt, \qquad \forall \sigma < \sigma_c^f. \tag{2.1}$$

Theorem 4. *If L_n and G_n are, respectively, the logarithmic and the generalized logarithmic mean functions of the n^{th} derivative $f^{(n)}$ of an entire function $f \in E$, then, for any $\delta \in R_+$, and $\sigma_1, \sigma_2 \in R$ such that $0 < \sigma_1 < \sigma_2 < \sigma_c^f$,*

$$L_n(\sigma_1, f^{(n)}) \leqslant \delta \, \frac{\exp(\delta\sigma_2)\, G_n(\sigma_2, f^{(n)}) - \exp(\delta\sigma_1)\, G_n(\sigma_1, f^{(n)})}{\exp(\delta\sigma_2) - \exp(\delta\sigma_1)} \leqslant L_n(\sigma_2, f^{(n)}). \tag{2.2}$$

Proof. We have, from (1.2),

$$G_n(\sigma, f^{(n)}) = \lim_{T \to +\infty} \frac{1}{2T \exp(\delta\sigma)} \int_0^{\sigma} \int_{-T}^{T} \log |f^{(n)}(x + it)| \exp(\delta x) \, dx \, dt$$

$$= \frac{1}{\exp(\delta\sigma)} \int_0^{\sigma} L_n(x, f^{(n)}) \exp(\delta x) \, dx.$$

Therefore

$$G_n(\sigma_1, f^{(n)}) = \frac{1}{\exp(\delta\sigma_1)} \int_0^{\sigma_1} L_n(x, f^{(n)}) \exp(\delta x) \, dx \tag{2.3}$$

and

$$G_n(\sigma_2, f^{(n)}) = \frac{1}{\exp(\delta\sigma_2)} \int_0^{\sigma_2} L_n(x, f^{(n)}) \exp(\delta x) \, dx. \tag{2.4}$$

From (2.3) and (2.4) we get

$$\left\{ \begin{aligned} & \exp(\delta\sigma_2)\, G_n(\sigma_2, f^{(n)}) - \exp(\delta\sigma_1)\, G_n(\sigma_1, f^{(n)}) \\ & = \int_{\sigma_1}^{\sigma_2} L_n(x, f^{(n)}) \exp(\delta x) \, dx \leqslant L_n(\sigma_2, f^{(n)}) \frac{1}{\delta} \left(\exp(\delta\sigma_2) - \exp(\delta\sigma_1)\right), \end{aligned} \right. \tag{2.5}$$

and

$$\exp(\delta\sigma_2)\, G_n(\sigma_2, f^{(n)}) - \exp(\delta\sigma_1)\, G_n(\sigma_1, f^{(n)}) \geqslant \geqslant L_n(\sigma_1, f^{(n)})\, \frac{1}{\delta}\left(\exp(\delta\sigma_2) - \exp(\delta\sigma_1)\right). \tag{2.6}$$

Combining (2.5) and (2.6), we get (2.2).

Theorem 5. *If L_n and G_n are, respectively, the logarithmic and the generalized logarithmic mean functions of the n^{th} derivative $f^{(n)}$ of an entire function $f \in E$ and M_n is the supremum function of $|f^{(n)}|$, then, for any $\delta \in R_+$,*

$$\limsup_{\sigma \to +\infty} \frac{G_n(\sigma, f^{(n)})}{\log M_n(\sigma, f^{(n)})} \leqslant \limsup_{\sigma \to +\infty} \frac{G_n(\sigma, f^{(n)})}{L_n(\sigma, f^{(n)})} \leqslant \frac{1}{\delta}. \tag{2.7}$$

Proof. We have, from (1.2),

$$G_n(\sigma, f^{(n)}) = \lim_{T \to +\infty} \frac{1}{2T \exp(\delta\sigma)} \int_0^{\sigma} \int_{-T}^{T} \log |f^{(n)}(x + it)| \exp(\delta x)\, dx\, dt$$

$$\leqslant \frac{1}{\delta} L_n(\sigma, f^{(n)})\left(1 - \exp(-\delta\sigma)\right),$$

therefore

$$\limsup_{\sigma \to +\infty} \frac{G_n(\sigma, f^{(n)})}{L_n(\sigma, f^{(n)})} \leqslant \frac{1}{\delta}. \tag{2.8}$$

Since, from (2.1), we get

$$L_n(\sigma, f^{(n)}) \leqslant \log M_n(\sigma, f^{(n)}),$$

it follows that

$$\frac{G_n(\sigma, f^{(n)})}{L_n(\sigma, f^{(n)})} \geqslant \frac{G_n(\sigma, f^{(n)})}{\log M_n(\sigma, f^{(n)})}, \tag{2.9}$$

whence, in view of (2.8),

$$\limsup_{\sigma \to +\infty} \frac{G_n(\sigma, f^{(n)})}{\log M_n(\sigma, f^{(n)})} \leqslant \limsup_{\sigma \to +\infty} \frac{G_n(\sigma, f^{(n)})}{L_n(\sigma, f^{(n)})} \leqslant \frac{1}{\delta}.$$

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References.

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[**2**] S. Bala, *Generalized logarithmic mean function of entire functions defined by Dirichlet series* (to appear.)

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