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On the Absolute Nörlund Summability of a Series Associated with a Fourier Series. (**)

In this paper absolute NÖRLUND summability of a series associated with a FOURIER series has been studied. These results include, as a special case, a theorem of the author [2], which is a generalization of a result of MOHANTY and MOHAPATRA [3].

1. – Let $f(t)$ be integrable (L) over $(-\pi, \pi)$ and periodic with period 2π and let

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t) .$$

Numbers x and s being fixed, we write

$$\varphi(t) = (1/2)\{f(x+t) + f(x-t) - 2s\} ,$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \varphi(u)\, \mathrm{d}u \qquad (\alpha > 0) ,$$

$$\varphi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \Phi_\alpha(t) , \qquad \varphi_0(t) = \varphi(t) ,$$

$$s_n = \sum_{k=0}^{n} A_k(x) .$$

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2. – Generalizing a result of MOHANTY and MOHAPATRA [3], the author [2] has recently obtained the following theorem.

Theorem A. If

$$\int_0^\pi \frac{|\varphi_\alpha(t)|}{t}\,dt < \infty \qquad (\alpha \geqslant 0), \tag{2.1}$$

then the series

$$\Sigma(s_n - s)/n \tag{2.2}$$

is summable $|C, \beta|$, $\beta > \alpha$.

The object of this Note is to study the corresponding problem for $|N, p_n|$ summability.

We prove the following theorems. In what follows we assume that $p_n \geqslant 0$.

Theorem 1. *Let $\{p_n\}$ be a non-increasing sequence of numbers such tha*

$$\{R_n\} \equiv \left\{\frac{(n+1)}{P_n}\,p_n\right\} \in \text{B.V.}, \tag{2.3}$$

$$\frac{P_k}{k^\alpha}\sum_{n=k}^{\infty}\frac{n^\alpha}{(n+1)P_n} \leqslant C \text{ (1)} \qquad (k = 1, 2, \ldots;\quad 0 \leqslant \alpha < 1). \tag{2.4}$$

If the condition (2.1) *holds, then the series* (2.2) *is summable* $|N, p_n|$ (2).

Theorem 2. *Suppose* (2.3) *holds and*

$$P_k \sum_{n=k}^{\infty}\frac{1}{(n+1)P_n} \leqslant C \qquad (k = 1, 2, \ldots), \tag{2.5}$$

$$\frac{n}{P_n}\sum_{k=0}^{n}|\Delta p_k| \leqslant C. \tag{2.6}$$

If (2.1) *holds,* $0 \leqslant \alpha < 1$, *then the series* (2.2) *is summable* $|N, p_n|$.

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(1) C is a constant not necessarily the same at each occurrence.
(2) We use here standard notations for NÖRLUND summability.

Theorem 3. *If*

$$\int_0^{\pi} \frac{|\varphi_1(t)|}{t}\, dt < \infty, \tag{2.7}$$

then the series (2.2) *is summable* $|N, p_n|$, *where* $\{p_n\}$ *is a non-decreasing sequence of numbers such that* (2.3) *holds and*

$$(P_k/k)\sum_{n=k}^{\infty}(1/P_n) \leqslant C \qquad (k = 1, 2, \ldots), \tag{2.8}$$

$$\{p_{n+1} - p_n\} \quad \text{is ultimately monotonic.} \tag{2.9}$$

3. – The following lemmas will be required for the proof of our theorems.

Lemma 1 [**1**]. *If* $\int_0^{\pi}(|\varphi_\alpha(t)|/t)\,dt < \infty$, *then* $\int_0^{\pi}(|\varphi_\beta(t)|/t)\,dt < \infty$, *where* $\beta > \alpha \geqslant 0$.

Lemma 2. *If* (2.7) *holds, then a necessary and sufficient condition for the series* (2.2) *to be summable* $|N, p_n|$ *is that*

$$\sum_{n=1}^{\infty} \frac{1}{n+1}\,|\sigma_n(x)| < \infty, \tag{3.1}$$

where $\sigma_n(x) = (1/P_{n-1})\sum_{k=1}^{n} p_{n-k}(s_k - s)$ *and the sequence* $\{p_n\}$ *satisfies the conditions* (2.3) *and* (2.5).

Proof of Lemma 2. Let

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}\frac{s_k - s}{k+1},$$

then

$$t_n - t_{n-1} = \frac{\sigma_n(x)}{n+1} + \frac{1}{(n+1)P_{n-1}}\sum_{k=0}^{n-1}\frac{R_k - R_n}{n-k}P_k(s_{n-k} - s).$$

Thus to establish the Lemma it is sufficient to show that

$$\sum_{1}^{\infty}\frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}\frac{R_k - R_n}{n-k}P_k(s_{n-k} - s)\right| < \infty. \tag{3.2}$$

Now, putting $D_k(t) = \{\sin(k+\frac{1}{2})t\}/\{2\sin(t/2)\}$, we have

$$\sum_{k=0}^{n-1} \frac{R_k - R_n}{n-k} P_k (s_{n-k} - s) = \frac{2}{\pi}\int_0^\pi \varphi(t) \sum_{k=0}^{n-1} \frac{R_k - R_n}{n-k} P_k D_{n-k}(t)\, \mathrm{d}t =$$

$$= \frac{1}{\pi} \Phi_1(\pi) \sum_{k=0}^{n-1} (-1)^{n-k} \frac{R_k - R_n}{n-k} P_k - \frac{2}{\pi}\int_0^\pi \frac{\varphi_1(t)}{t} \sum_{k=0}^{n-1} \frac{R_k - R_n}{n-k} P_k \left\{ t^2 \frac{\mathrm{d}}{\mathrm{d}t} D_{n-k}(t) \right\} \mathrm{d}t\, .$$

It is therefore sufficient to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \frac{1}{P_{n-1}} \left| \sum_{k=0}^{n-1} (-1)^k \frac{R_k - R_n}{n-k} P_k \right| < \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \sum_{k=0}^{n-1} \frac{R_k - R_n}{n-k} P_k \left\{ t^2 \frac{\mathrm{d}}{\mathrm{d}t} D_{n-k}(t) \right\} \right| < \infty, \tag{3.4}$$

uniformly in $0 < t < \pi$.

Proof of (3.3). We have on applying ABEL's transformation

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \sum_{k=0}^{n-1} (-1)^k \frac{(R_k - R_n)P_k}{n-k} \right| =$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \sum_{k=0}^{n-1} \Delta R_k \sum_{\nu=0}^{k} (-1)^\nu \frac{P_\nu}{n-\nu} \right| \leqslant$$

$$\leqslant \sum_{k=1}^{\infty} |\Delta R_k| P_k \sum_{n=k+1}^{\infty} \frac{1}{(n+1)P_{n-1}} < \sum_{k=1}^{\infty} |\Delta R_k| < \infty,$$

by virtue of the conditions (2.3) and (2.5).

Proof of (3.4). Let $m = [n/2]$ and $\tau = [\pi/t]$. Then the left hand side expression of (3.4) is

$$\leqslant \sum_{n \leqslant \tau} \frac{1}{(n+1)P_{n-1}} \left| \sum_{k=0}^{m-1} \cdots \right| + \sum_{n > \tau} \frac{1}{(n+1)P_{n-1}} \left| \sum_{k=0}^{m-1} \cdots \right| +$$

$$+ \sum_{n} \frac{1}{(n+1)P_{n-1}} \left| \sum_{k=0}^{n-1} \cdots \right| = \Sigma_1 + \Sigma_2 + \Sigma_3, \qquad \text{say.}$$

Now

$$\Sigma_1 \leqslant C \sum_{n \leqslant \tau} \frac{1}{(n+1)P_{n-1}} \sum_{k=0}^{m-1} t P_k \leqslant t \sum_{n \leqslant T} 1 = O(1) .$$

Applying ABEL's transformation to the inner sum of Σ_2 and using the facts that $\{P_k/(n-k)\}$ is monotonic non-decreasing with respect to k for $k<n$ and

$$t^2 \sum_{\nu=0}^{k} \frac{\mathrm{d}}{\mathrm{d}t} D_{n-\nu}(t) = O(n) + O(1/t) ,$$

we have

$$\Sigma_2 \leqslant \sum_{n>\tau} \frac{1}{(n+1)P_{n-1}} \sum_{k=0}^{m-2} |\Delta R_k| \frac{P_k}{n-k} \{O(n) + O(1/t)\} +$$

$$+ \sum_{n>T} \frac{1}{(n+1)P_{n-1}} |R_{m-1} - R_n| \frac{P_{m-1}}{n-m+1} \{O(n) + O(1/t)\}$$

$$= O\left(\sum_{n>T} \frac{1}{(n+1)P_{n-1}} \sum_{k=0}^{n-1} P_k |\Delta R_k| \right) + O\left(\sum_{n>T} \frac{1}{n+1} \left| \sum_{\nu=m-1}^{n-1} \Delta R_\nu \right| \right)$$

$$= O(1) + O\left(\sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{\nu=m-1}^{n-1} |\Delta R_\nu| \right) = O(1) + O\,(\sum_{\nu=1}^{\infty} |\Delta R_\nu|) = O(1) .$$

Again

$$\Sigma_3 \leqslant \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \sum_{k=m}^{n-1} |\Delta R_k| \cdot \left| \sum_{\nu=m}^{k} \frac{P_\nu}{n-\nu} t^2 \frac{\mathrm{d}}{\mathrm{d}t} D_{n-\nu}(t) \right|$$

$$\leqslant C \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \sum_{k=m}^{n-1} P_k |\Delta R_k| \leqslant C \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=m}^{n-1} |\Delta R_k| = O(1) .$$

This completes the proof of the Lemma.

Lemma 3. *Let*

$$g(n, t) = \frac{2}{\pi P_{n-1}} \sum_{k=1}^{n} p_{n-k} D_k(t) .$$

Then, for all t and all non-negative sequence $\{p_n\}$,

$$g(n, t) = O(n), \qquad \frac{d}{dt} g(n, t) = O(n^2) .$$

Also:

(a) *if* $\{p_n\}$ *is non-increasing sequence of non-negative numbers, then*

$$g(n, t) = O\left(\frac{P_{[1/t]}}{t P_{n-1}}\right), \qquad \frac{d}{dt} g(n, t) = O\left(\frac{n P_{[1/t]}}{t P_{n-1}}\right) ;$$

(b) *if* $\{p_n\}$ *is non-negative sequence satisfying* (2.3) *and* (2.6), *then*

$$g(n, t) = O\left(\frac{1}{n t^2}\right), \qquad \frac{d}{dt} g(n, t) = O\left(\frac{1}{t^2}\right) .$$

The proof is quite simple.

Lemma 4. *Let*

$$F(n, u) = \int_u^{\pi} (t-u)^{-\alpha} \frac{d}{dt} g(n, t)\, dt , \qquad 0 \leqslant \alpha < 1 .$$

If condition (a) *of Lemma* 3 *is satisfied, then*

$$F(n, u) = \begin{cases} O(n^{\alpha+1}) , & 0 < u \leqslant n^{-1} \\ O\left(\dfrac{n^{\alpha} P_{[1/u]}}{u P_{n-1}}\right) , & n^{-1} < u < \pi , \end{cases}$$

while, if condition (b) *of Lemma* 3 *is satisfied,*

$$F(n, u) = \begin{cases} O(n^{\alpha+1}) , & 0 < u \leqslant n^{-1}, \\ O(n^{\alpha-1}/u^2) , & n^{-1} < u < \pi . \end{cases}$$

Proof of Lemma 4. Let

$$F(n, u) = \int_u^{u+(1/n)} \dots + \int_{u+(1/n)}^{\pi} \dots .$$

Applying mean value theorem and Lemma 3, the result follows.

4. - Proof of Theorem 1. By virtue of Lemmas 1 and 2 and the fact that (2.4) $\Rightarrow$ (2.5), it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \, |\sigma_n(x)| < \infty .$$

Now

$$\sigma_n(x) = \int_0^{\pi} \varphi(t) \, g(n, t) \, \mathrm{d}t = [\Phi_1(t) \, g(n, t)]_0^{\pi} - \int_0^{\pi} \Phi_1(t) \, \frac{\mathrm{d}}{\mathrm{d}t} \, g(n, t) \, \mathrm{d}t =$$

$$= \frac{C}{P_{n-1}} \sum_{k=1}^{n} (-1)^k p_{n-k} - \frac{1}{\Gamma(1-\alpha)} \int_0^{\pi} \Phi_\alpha(u) \, \mathrm{d}u \int_u^{\pi} (t-u)^{-\alpha} \, \frac{\mathrm{d}}{\mathrm{d}t} \, g(n, t) \, \mathrm{d}t .$$

Thus in view of (2.1) it is sufficient to prove that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \, \Big| \sum_{k=1}^{n} (-1)^k p_{n-k} \Big| < \infty , \tag{4.1}$$

$$\sum_{n=1}^{\infty} \frac{u^{\alpha+1}}{n+1} \, |F(n, u)| < \infty , \tag{4.2}$$

uniformly in $0 < u < \pi$.

Proof of (4.1). We have

$$\sum_{k=1}^{n} (-1)^k p_{n-k} = O\Big(\sum_{\nu=0}^{n-2} |\Delta p_\nu|\Big) + O(1)$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left|\sum_{k=1}^{n}(-1)^k p_{n-k}\right| = O\left(\sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \sum_{\nu=0}^{n-2}|\Delta p_\nu|\right) + O(1)$$

$$= O\left(\sum_{\nu=1}^{\infty}|\Delta p_\nu| \sum_{n=\nu+2}^{\infty} \frac{1}{(n+1)P_{n-1}}\right) + O(1) = O\left(\sum_{\nu=1}^{\infty} \frac{|\Delta p_\nu|}{P_{\nu+1}}\right) + O(1) = O(1),$$

by virtue of the condition (2.3).

Proof of (4.2). We have

$$\sum_{n=1}^{\infty} \frac{u^{\alpha+1}}{n+1} |F(n, u)| \leqslant C \sum_{n\leqslant u^{-1}} \frac{u^{\alpha+1} n^{\alpha+1}}{n+1} + C \sum_{n>u^{-1}} \frac{u^{\alpha+1}}{n+1} \cdot n^\alpha \frac{P_{[1/u]}}{u P_{n-1}}$$

$$\leqslant C + C u^\alpha P_{[1/u]} \sum_{n=[1/u]+1}^{\infty} \frac{n^\alpha}{(n+1)P_{n-1}} = O(1)$$

uniformly in $0 < u < \pi$, by virtue of the condition (2.4).

This completes the proof of Theorem 1.

5. – Proof of Theorem 2. In the proof of Theorem 1 upto (4.1) we have used only two conditions namely (2.3) and (2.5) of Theorem 2. It is therefore sufficient to prove (4.2). Applying Lemma 4 we have

$$\sum_{n=1}^{\infty} \frac{u^{\alpha+1}}{n+1} |F(n, u)| \leqslant C \sum_{n\leqslant u^{-1}} \frac{u^{\alpha+1}}{n+1} n^{\alpha+1} + C \sum_{n>u^{-1}} \frac{u^{\alpha+1}}{n+1} \frac{n^{\alpha-1}}{u^2} =$$

$$= O(1) + O\left(\sum_{n>u^{-1}} u^{\alpha-1} n^{\alpha-2}\right) = O(1),$$

uniformly in $0 < u < \pi$.

This proves Theorem 2.

6. – Proof of Theorem 3. It is obvious that (2.8) $\Rightarrow$ (2.5) and hence by virtue of Lemma 2 it is sufficient to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n+1} |\sigma_n(x)| < \infty.$$

As in the proof of Theorem 1

$$\sigma_n(x) = \frac{C}{P_{n-1}} \sum_{k=1}^{n} (-1)^k p_{n-k} - \int_0^{\pi} \Phi_1(t) \frac{d}{dt} g(n, t)\, dt .$$

Since (4.1) is true under the conditions (2.3) and (2.8), it is sufficient by virtue of the hypothesis to prove that

$$\sum_{n=1}^{\infty} \frac{t^2}{n+1} \left| \frac{d}{dt} g(n, t) \right| < \infty \tag{6.1}$$

uniformly in $0 < t < \pi$. We have

$$\frac{d}{dt} g(n, t) = \frac{2}{\pi P_{n-1}} \sum_{k=1}^{n} (p_{n-k} - p_{n-k-1}) \sum_{\nu=0}^{k} \frac{d}{dt} D_\nu(t) .$$

Writing the sum in (6.1) as

$$\Sigma = \sum_{n \leqslant 1/t} \ldots + \sum_{n > 1/t} \ldots = L_1 + L_2 ,$$

we have

$$L_1 \leqslant C \sum_{n \leqslant 1/t} \frac{t^2}{(n+1) P_{n-1}} \sum_{k=1}^{n} (p_{n-k} - p_{n-k-1}) \frac{n^2}{t} \leqslant C t \sum_{n \leqslant 1/t} \frac{n p_{n-1}}{P_{n-1}} \leqslant C .$$

Let m_0 be a constant such that $\{p_n - p_{n-1}\}$ is monotonic for $n > m_0$. Then

$$L_2 \leqslant C \sum_{n > 1/t} \frac{t^2}{(n+1) P_{n-1}} \left| \sum_{k=1}^{n-m_0-1} (p_{n-k} - p_{n-k-1}) \sum_{\nu=0}^{k} \frac{d}{dt} D_\nu(t) \right| +$$

$$+ C \sum_{n > 1/t} \frac{t^2}{(n+1) P_{n-1}} \left| \sum_{k=n-m_0}^{n} (p_{n-k} - p_{n-k-1}) \sum_{\nu=0}^{k} \frac{d}{dt} D_\nu(t) \right|$$

$$= L_{21} + L_{22} , \qquad \text{say.}$$

By virtue of the fact that

$$\sum_{n=1}^{\infty} (1/P_n) < \infty ,$$

we observe that

$$L_{22} \leqslant C \sum_{n>1/t} \frac{t^2}{(n+1)P_{n-1}} \frac{n}{t^2} \leqslant C \sum_{n>1/t} \frac{1}{P_{n-1}} \leqslant C\,.$$

We now proceed to show that $L_{21} \leqslant C$.

Case (i). Let $\{p_n - p_{n-1}\}$ be monotonic non-decreasing for $n > m_0$. Since

$$\sum_{k=1}^{n-m_0-1} (p_{n-k} - p_{n-k-1}) \sum_{\nu=0}^{k} \frac{\mathrm{d}}{\mathrm{d}t} D_\nu(t) = O\left(\frac{n}{t^3}\right) (p_n - p_{n-1})\,, \tag{6.2}$$

we have

$$L_{21} = O\left(\sum_{n>1/t} \frac{t^2}{(n+1)P_{n-1}} \frac{n}{t^3} (p_n - p_{n-1})\right) = O\left(\frac{1}{t} \sum_{n>1/t} \frac{p_n - p_{n-1}}{P_{n-1}}\right) =$$

$$= O\left(\frac{1}{t} \sum_{n>1/t} \frac{1}{n} \left| \Delta\left(\frac{n p_{n-1}}{P_{n-1}}\right) \right| \right) + O\left(\frac{1}{t} \sum_{n>1/t} \frac{1}{n^2}\right) = O(1)\,.$$

Case (ii). Suppose $\{p_n - p_{n-1}\}$ is monotonic non-increasing. In this case the expression in (6.2) is $O((n/t^2) p_{[1/t]})$. Hence

$$L_{21} = O\left(\sum_{n>1/t} \frac{t^2}{(n+1)P_{n-1}} \frac{n}{t^2} p_{[1/t]}\right) =$$

$$= O\left(p_{[1/t]} \sum_{n>1/t} \frac{1}{P_{n-1}}\right) = O\left(\frac{[1/t]\, p_{[1/t]}}{P_{[1/t]}}\right) = O(1)\,,$$

uniformly in $0 < t < \pi$.

This proves Theorem 3.

7. – It is evident that if $\{p_n\}$ is non-decreasing sequence satisfying (2.3), then (2.6) holds. Also (2.4) $\Rightarrow$ (2.5). Thus we deduce the following theorem.

Theorem 4. *Let $\{p_n\}$ be any monotonic sequence of non-negative numbers such that* (2.3) *and* (2.4) *hold. If* (2.1) *holds, then the series* (2.2) *is summable* $|\mathrm{N}, p_n|$.

It is clear from our theorems that summability $|\mathrm{N}, p_n|$ of $\Sigma(s_n - s)/n$ is a local property of the generating function.

References.

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