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Absolute Riesz Summabitity Factors of Fourier Series. (**)

1. - Definitions and notations.

Let $\lambda = \lambda(w)$ be a differentiable, monotonic increasing, function of w, tending to infinity with w. For a given infinite series $\sum a_n$, we write

$$A_r(w) = \sum_{n \leqslant w} \{\lambda(w) - \lambda(n)\}^r a_n \qquad (r \geqslant 0) .$$

The series $\sum a_n$ is summable $|R, \lambda, r|$, $r \geqslant 0$, if

$$\int_A^\infty |d(A_r(w)/\{\lambda(w)\}^r)| < \infty ,$$

where A is a positive number. (1)

Now, for $r > 0$, $m < w < m+1$,

$$\frac{d}{dw}[A_r(w)/\{\lambda(w)\}^r] = \frac{r\,\lambda'(w)}{\{\lambda(w)\}^{r+1}} \sum_{n \leqslant w} \{\lambda(w) - \lambda(n)\}^{r-1} \lambda(n)\, a_n .$$

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(1) OBRECHKOFF [6], [7].

Hence, the series $\sum a_n$ is said to be summable $|R, \lambda, r|$ $(r > 0)$, if

$$\int_A^\infty \frac{r\,\lambda'(w)}{\{\lambda(w)\}^{r+1}}\,\Big|\sum_{n\leqslant w}\{\lambda(w)-\lambda(n)\}^{r-1}\lambda(n)\,a_n\Big|\,dw$$

is convergent.

Evidently summability $|R, \lambda, 0|$ is equivalent to absolute convergence.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality the constant term in the FOURIER series of $f(t)$ can be taken to be zero, so that

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t)\,,$$

and

$$\int_{-\pi}^{\pi} f(t)\,dt = 0\,.$$

Throughout this paper we use the following notations:

(1.1) $$\in BV(a, b) = \text{is of bounded variation in } (a, b)\,,$$

(1.2) $$\varphi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}\,,$$

(1.3) $$\varphi_\alpha(t) = \alpha t^{-\alpha}\int_0^t (t-u)^{\alpha-1}\varphi(u)\,du \qquad (\alpha > 0)\,,$$

(1.4) $$\eta(n) = (\log(n+1))^{-2}\,(\log\log(n+2))^{-1},$$

(1.5) $$e(w) = \exp\{\log w\,\log\log w\}\,,$$

(1.6) $$E(w, t) = \sum_{n\leqslant w} e(n)\,\eta(n)\cos nt\,,$$

(1.7) $$G(w, t) = \int_0^t u\left(\log\log\frac{k}{u}\right)^{-1}\frac{\partial}{\partial u}E(w, u)\,du\,,$$

(1.8) $$H(w, t) = \int_t^\pi u\left(\log\log\frac{k}{u}\right)^{-1}\frac{\partial}{\partial u}E(w, u)\,du\,.$$

2. - Introduction.

Generalizing earlier theorems of MOHANTY (cf. [**5**]) and himself (cf. [**8**]), in 1957 PATI (cf. [**9**]) established the following theorem.

Theorem A. If α is an integer $\geqslant 1$ and $\varphi_\alpha(t)\log(k/t) \in \mathrm{BV}(0, \pi)$, then the FOURIER series of $f(t)$, at $t = x$ is, summable $|\mathrm{R}, \exp\{(\log w)^{1+(1/\alpha)}\}, \alpha + \delta|$, for every $\delta > 0$.

Extending Theorem A, SINHA (cf. [**10**]) proved the following theorem for the case of integral α, where $\alpha \geqslant 1$. For the case of general positive α, the Theorem B is due to MATSUMOTO [**4**], and has been generalised later, by MALVIYA [**3**], by giving a wrong proof of Lemma 3.

Theorem B. If $\alpha > 0$, $\beta \geqslant 0$ and $\varphi_\alpha(t)\{\log(k/t)\}^{\alpha\beta} \in \mathrm{BV}(0, \pi)$, where $k > \pi\exp(1 + \alpha\beta)$, then the FOURIER series of $f(t)$, at $t = x$, is summable $|\mathrm{R}, \exp\{(\log w)^{1+\beta}\}, \alpha + \delta|$, for every $\delta > 0$.

In 1961, DIKSHIT investigated the summability factors ε_n which can make the series $\sum A_n(x)\varepsilon_n$ summable $|\mathrm{R}, \exp\{(\log w)^{1+\beta}\}, \alpha|$, $\alpha \geqslant 0$, whenever the condition $\varphi_\alpha(t)\{\log(k/t)\}^{\alpha\beta} \in \mathrm{BV}(0, \pi)$ is satisfied.

The present author [**1**] proved the following theorem:

Theorem C. If $\varphi_1(t)\log\log(k/t) \in \mathrm{BV}(0, \pi)$, then the FOURIER series of $f(t)$, at $t = x$, is summable $|\mathrm{R}, e(w), 1 + \delta|$ $(\delta > 0)$.

In particular, taking $\alpha = \beta = 1$, and replacing $\log(k/t)$ by $\log\log(k/t)$ in Theorem B, we answer the question: what possible factors, under the same restrictions as in Theorem C with order 1 in place of $1 + \delta$ in Theorem C, can be obtained to make the series $\sum A_n(x)\eta(n)$ summable $|\mathrm{R}, e(w), 1|$. We pricisely prove the following

Theorem. *If* $\varphi_1(t)\log\log(k/t) \in \mathrm{BV}(0, \pi)$, *where* $k \geqslant \pi e^2$, *then* $\sum_{n=1}^{\infty} A_n(x)\eta(n)$ *is summable* $|\mathrm{R}, e(w), 1|$.

3. – We require the following order-estimates for the proof of the theorem:

$$E(w, t) = O\{w\ e(w)\ \eta(w)/(\log\log w + 1)\}, \tag{3.1}$$

$$E(w, t) = O\{t^{-1}\ e(w)\ \eta(w)\}, \tag{3.2}$$

$$G(w, t) = O\left\{\frac{t}{\log\log(k/t)}\ w\ e(w)\ \eta(w)\ (\log\log w + 1)^{-1}\right\}, \tag{3.3}$$

$$H(w, t) = O\{\log(k/t)\ e(w)\ \eta(w)\}. \tag{3.4}$$

Proof of (3.1). Let $m \leqslant w < m+1$, and let $\lambda(1)=1$ and $\lambda(n)=e(n)$, for $n \geqslant 2$. Then

$$E(w,t) = \sum_{n=1}^{m} e(n)\ \eta(n)\ \cos nt$$

$$= O\{\sum_{n=1}^{m} e(n)\ \eta(n)\}$$

$$= O(1) + O\{\sum_{n=q}^{m} e(n)\ \eta(n)\}\ ,$$

where q is an integer such that

$$\sum_{n=q}^{m} e(n)\ \eta(n) < \int_{q}^{m} e(x)\ \eta(x)\ \mathrm{d}x + e(m)\ \eta(m)\ ,$$

so that

$$\sum_{n=q}^{m} e(n)\ \eta(n) =$$

$$= O\left\{w\ \eta(w)\ (\log\log w+1)^{-1} \int_{q}^{w} \frac{\log\log x+1}{x}\ e(x)\,\mathrm{d}x\right\} + O\{e(w)\ \eta(w)\}$$

$$= O\{w\ e(w)\ \eta(w)\ (\log\log w+1)^{-1}\} + O\{e(w)\ \eta(w)\}$$

$$= O\{w\ e(w)\ \eta(w)\ (\log\log w+1)^{-1}\}\ .$$

Hence, finally, we have

$$E(w,t) = O\{w\ e(w)\ \eta(w)\ (\log\log w+1)^{-1}\}\ .$$

Proof of (3.2). Let $m \leqslant w < m+1$, and let $\lambda(1)=1$ and $\lambda(n)=e(n)$, for $n \geqslant 2$. Then

$$E(w,t) = \sum_{n=1}^{m} e(n)\ \eta(n)\ \cos nt$$

$$= (\sum_{n=1}^{p-1} + \sum_{p}^{m})(e(n)\,\eta(n)\ \cos nt)$$

$$= R + S\ , \qquad \text{say,}$$

where p is an integer such that $e(n)\,\eta(n)$ is steadily increasing for $n \geqslant p$.

Now, we have $R = O(1)$ and, by ABEL's lemma, we have

$$S = e(m)\ \eta(m)\ \left|\sum_{n=p'}^{m} \cos nt\right| \qquad (p \leqslant p' \leqslant m)$$

$$= O\{t^{-1}\ e(w)\ \eta(w)\}\,.$$

Proof of (3.3). By the second mean value theorem, we have

$$G(w,t) = t\left(\log\log\frac{k}{t}\right)^{-1}\int_{t'}^{t}\frac{\partial}{\partial u}E(w,u)\ du \qquad (0 < t' < t)$$

$$= t\left(\log\log\frac{k}{t}\right)^{-1}(E(w,t) - E(w,t'))$$

$$= O\left\{t\left(\log\log\frac{k}{t}\right)^{-1} w\ e(w)\ \eta(w)\ (\log\log w + 1)^{-1}\right\},$$

by (3.1).

Proof of (3.4). Integrating by parts, we have

$$H(w,t) = \left[v\left(\log\log\frac{k}{v}\right)^{-1}E(w,v)\right]_t^{\pi}$$

$$- \int_t^{\pi}\left\{\left(\log\log\frac{k}{v}\right)^{-1} + \left(\log\frac{k}{v}\right)^{-1}\left(\log\log\frac{k}{v}\right)^{-2}\right\}E(w,v)\,dv$$

$$= O\left\{\left(\log\log\frac{k}{t}\right)^{-1}e(w)\ \eta(w)\right\} + O\left\{e(w)\ \eta(w)\int_t^{\pi}v^{-1}\left(\log\log\frac{k}{v}\right)^{-1}dv\right\} \qquad \text{(by (3.2))}$$

$$= O\left\{\log\frac{k}{t}\ e(w)\ \eta(w)\right\}.$$

4. – For the proof of the theorem we shall require the following lemmas:

Lemma 1 (2). *If* $\sum a_n$ *is summable* $|\mathrm{R}, \lambda_n, r|$, $r \geqslant 0$, *then it is also summable* $|\mathrm{R}, \lambda_n, r'|$, $r' > r$.

Lemma 2. *The Fourier series of the even function* $(\log\log|k/t|)^{-1}$ $(k \geqslant \pi e^2)$, *defined outside* $(-\pi, \pi)$ *by periodicity is absolutely convergent at* $t = 0$.

Proof. Let

$$(\log\log|k/t|)^{-1} \sim \sum \alpha_n \cos n\,t\,,$$

where

$$\alpha_n = \frac{2}{\pi}\int_0^{\pi} (\cos nt/\log\log(k/t))\,dt$$

$$= O\{n^{-1}(\log n)^{-1}(\log\log n)^{-2}\}\,,$$

by using the arguments used in Mohanty ([5], lemma 6). And hence the Lemma follows.

Lemma 3. *The Fourier series of the even function* $(\log|k/t|)^{-1}.$ $(\log\log|k/t|)^{-2}$, *defined outside* $(-\pi, \pi)$ *by periodicity, is absolutely convergent, at* $t = 0$.

Proof. Proof is parallel to that used by Mohanty in his lemma 6 of [5].

Lemma 4. *The integral*

$$I = \int_{e^2}^{\infty} e^{-1}(w)\; w^{-1}\,(\log\log w + 1)^{-1}\,|G(w, \pi)|\;\, dw < \infty\,.$$

Proof. Integrating by parts, we have

$$G(w, \pi) = \pi\left(\log\,\log\frac{k}{\pi}\right)^{-1} E(w, \pi) - \int_0^{\pi} E(w, u)\,\frac{\partial}{\partial u}\left(u\left(\log\,\log\frac{k}{u}\right)^{-1}\right)du$$

$$= O\{e(w)\;\eta(w)\} + O\{\sum_{n \leqslant w} e(n)\;\eta(n)\;\beta_n'\}, \qquad \text{(by (3.2))};$$

(2) Obrechkoff [6], [7].

where

$$\beta'_n = \int_0^\pi \left[\left(\log\log\frac{k}{t}\right)^{-1} + \left(\log\frac{k}{t}\right)^{-1} \left(\log\log\frac{k}{t}\right)^{-2} \right] \cos nt \, dt =$$

$$= \frac{\pi}{2} (\alpha_n + \beta_n) ,$$

α_n and β_n are as defined in Lemma 2 and Lemma 3, respectively. Hence by Lemma 2 and Lemma 3, we have

$$\beta'_n = O\{n^{-1} (\log(n+1))^{-1} (\log\log(n+2))^{-2}\} .$$

Therefore

$$I = O\{ \int_{e^2}^\infty w^{-1} (\log w)^{-2} (1 + (\log\log w)^{-1}) \, dw\} +$$
$$+ O\{ \int_{e^2}^\infty e^{-1}(w) \, w^{-1} (\log\log w + 1) \, | \sum_{n \leqslant w} e(n) \, \eta(n) \, \beta_n | \, dw\}$$
$$= O(1) ,$$

by the convergence of the first integral and, since $\sum_{n=1}^\infty \eta(n)\, \beta_n$ is absolutely convergent, the second integral is also convergent by Lemma 1. Hence the proof of the Lemma follows.

5. - Proof of the Theorem. Since,

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \varphi(t) \cos nt \, dt ,$$

integrating by parts and using the fact $\varphi_1(\pi) = 0$, we have

$$A_n(x) = \frac{2}{\pi} \int_0^\pi n t \, \varphi_1(t) \, \sin nt \, dt$$

$$= -\frac{2}{\pi} \int_0^\pi d \left\{ \varphi_1(t) \log\log\frac{k}{t} \right\} \int_0^t \left(nv \, \sin nv / \log\log\frac{k}{v} \right) dv$$

(integrating by parts).

The series $\sum_{n=1}^{\infty} A_n(x)\ \eta(n)$ is summable $|\mathrm{R}, e(w), 1|$ if

$$I = \frac{4}{\pi}\int_{e^2}^{\infty} e^{-1}(\omega)\ w^{-1}\ (\log\log w+1)\left|\sum_{n\leqslant w} e(n)\ \eta(n)\cdot\right.$$

$$\left.\cdot\int_0^{\pi} \mathrm{d}\left\{\varphi_1(t)\log\log\frac{k}{t}\right\}\int_0^t \left(nv\ \sin nv/\log\log\frac{k}{v}\right)\mathrm{d}v\right|\mathrm{d}w,$$

is convergent. But

$$I \leqslant \frac{4}{\pi}\int_0^{\pi}\left|\mathrm{d}\left\{\varphi_1(t)\log\log\frac{k}{t}\right\}\right|\int_{e^2}^{\infty} e^{-1}(w)\ w^{-1}\ (\log\log w+1)\,|G(w, t)|\,\mathrm{d}w\,.$$

Therefore, since

$$\int_0^{\pi}\left|\mathrm{d}\left\{\varphi_1(t)\ \log\log\frac{k}{t}\right\}\right| < \infty\,,$$

for the proof of the Theorem, it is sufficient to show that

$$J = \int_{e^2}^{\infty} e^{-1}(w)\ w^{-1}\ (\log\log w+1)\,|G(w, t)|\,\mathrm{d}w = O(1)\,,$$

uniformly in $0 < t < \pi$.

On writing $\tau = k/t$,

$$J = \int_{e^2}^{\tau}\ldots + \int_{\tau}^{\infty}\ldots = J_1 + J_2\,, \qquad \text{say.}$$

By (3.3),

$$J_1 = O(1)\,.$$

And using the fact

$$G(w, t) = G(w, \pi) - H(w, t)\,,$$

we have

$$J_2 \leqslant \int_{\tau}^{\infty} e^{-1}(w)\ w^{-1}\ (\log\log w+1)\,|G(w, \pi)|\,\mathrm{d}w +$$

$$+ \int_{\tau}^{\infty} e^{-1}(w)\ w^{-1}\ (\log\log w+1)\,|H(w, t)|\,\mathrm{d}w = J_{21} + J_{22}\,, \qquad \text{say.}$$

Now

$$J_{2,1} \leqslant \int_{e^2}^{\infty} e^{-1}(w)\, w^{-1}\, (\log\log w+1)\, |G(w, \pi)|\, dw = O(1)\,,$$

by Lemma 4. And, by (3.4),

$$J_{2,2} = O\left\{\log \frac{k}{t} \int_{\tau}^{\infty} w^{-1} (\log\, w)^{-2}\, \left(1+(\log\log w)^{-1}\right) dw\right\} = O(1)\,,$$

uniformly in $0 < t < \pi$.

This completes the proof of the Theorem.

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