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Fixed Point Theorems in Generalized Metric Space. (**)

The concept of generalized complete metric space, first introduced by W. A. J. Luxemburg [5], has been of continuing interest in recent years. Two contraction mapping theorems were given by Luxemburg on such a space and then applied to the theory of ordinary differential equations. These theorems have since been generalized to a family of contractions by such mathematicians as Monna [7], Edelstein [4], and Margolis [6]. Further generalizations will be given in the present paper.

Definition. Let X be a non empty set. If there is defined a distance function on $X \times X$ such that d: $X \times X \to [0, \infty]$, satisfying the following conditions:

(D$_1$) $d(x, y) = 0$ if and only if $x = y$,

(D$_2$) $d(x, y) = d(y, x)$,

(D$_3$) $d(x, y) \leqslant d(x, z) + d(z, y)$,

(D$_4$) $\lim\limits_{n, m \to \infty} d(x_n, x_m) = 0 \quad \Rightarrow \quad \lim\limits_{n \to \infty} (x, x_n) = 0$,

where $x_n \in X$ $(n = 1, 2, 3, ...)$ and x is unique;

then X, with the metric d, i.e. (X, d), is called a generalized complete metric space.

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Examples of such a space would be the extended real line with the usual metric and the extended complex plane with the usual metric.

Theorem 1. *Let f^p (p is any positive integer) be a mapping of the generalized complete metric space X into itself satisfying the following conditions:*

(a) *There exists a constant q $(0 < q < 1)$ such that $d(f^p x, f^p y) \leqslant q\, d(x, y)$ for all $x, y \in X$ with $d(x, y) < \infty$.*

(b) *For every sequence of successive approximations $x_n = f^p x_{n-1}$ $(n = 1, 2, \ldots)$, where x_0 is an arbitrary element of X, there exists an index $N(x_0)$ such that $d(x_N, x_{N+r}) < \infty$ for all $r = 1, 2, \ldots$.*

(c) *If x and y are two fixed points of f^p, i.e. $f^p(x) = x$, and $f^p(y) = y$, then $d(x, y) < \infty$.*

Then f has a unique fixed point $x = \lim_{n\to\infty} x_n$.

Proof. Let $x_0 \in X$ and form the sequence $x_n = f^p x_{n-1}$ $(n = 1, 2, \ldots)$. By (b) there exists and index $N(x_0)$ such that

$$d(x_N, x_{N+r}) < \infty \qquad (r = 1, 2, \ldots).$$

Hence by (b) we have $d(x_n, x_{n+r}) < \infty$ for $n \geqslant N$ and $r = 1, 2, \ldots$. Then (a) implies that $d(x_{N+1}, x_{N+2}) \leqslant q\, d(x_N, f^p x_N)$ and generally

$$d(x_n, x_{n+1}) \leqslant q^{n-N} d(x_N, f^p x_N) \quad \text{for} \quad n \geqslant N.$$

Since by (D_3) we have $d(x_n, x_{n+r}) \leqslant \sum_{i=1}^{\gamma} d(x_{n+i}, x_{n+i-1})$, we obtain by the above inequality

$$d(x_n, x_{n+r}) \leqslant \{q^{n-N}(1 - q^r)/(1 - q)\}\, d(x_N, f^p x_N) \qquad (n \geqslant N;\, r = 1, 2, \ldots).$$

Hence x_n is a d-Cauchy sequence. From (D_4) it follows then that there exists an element $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. For this element x we conclude by (D_3) that

$$d(x, f^p x) \leqslant d(f^p x, x_n) + \big(d(x_n, x) \leqslant q\, d(x, x_{n-1}) + d(x_n, x)\big) \qquad \text{for } n \geqslant N.$$

Hence $d(x, f^p x) = 0$ and, by (D_1), $f^p x = x$. So x is a fixed point of f^p. Assume now that $f^p y = y$ with $x \neq y$. Then, by (D_3), $d(x, y) < \infty$ and by (a) we get

$$0 \leqslant d(x, y) = d(f^p x, f^p y) \leqslant q\; d(x, y).$$

This implies that $d(x, y) = 0$ and hence $x = y$. Therefore x is a unique fixed point of f^p, and hence a unique fixed point of f.

Remark. In case $p = 1$, we get a well-known theorem of LUXEMBURG [5].

Now we generalize a theorem of LUXEMBURG [5] for a family of contraction mappings in the following way:

Theorem 2. *Suppose f_i $(i = 1, 2, ...)$ is a sequence of self mappings of a generalized complete metric space X satisfying the following conditions:*

(1) *There exists a constant k $(0 < k < 1)$ such that $d(f_i x, f_i y) < k\, d(x, y)$ for all $x, y \in X$ with $d(x, y) < \infty$.*

(2) *$f_i f_j = f_j f_i$ i.e. any two mappings commute.*

(3) *For every sequence $x_n = f_i x_{n-1}$ $(n = 1, 2, ...)$, where x_0 is an arbitrary element of X, there exists an index $N(x_0)$ such that $d(x_N, x_{N+r}) < \infty$ for all $r = 1, 2, ...,$ and $i = 1, 2, ...$.*

(4) *If x, y are any two fixed points of the mapping f_i, then $d(x, y) < \infty$ $(i = 1, 2, ...)$.*

Then the sequence f_i has a common unique fixed point.

Proof. Conditions (1), (3) and (4) ensure that each mapping f_i $(i = 1, 2, ...)$ will have a unique fixed point. Assume now $f_i(x_i) = x_i$, and $f_j(x_j) = x_j$ $(x_i \neq x_j)$. Since the family f_i commutes, we have

$$f_i(f_j(x_i)) = f_j(f_i(x_i)) = f_j(x_i)\,.$$

Hence $f_j(x_i)$ is a fixed point of f_i. But f_i has a unique fixed point x_i. Therefore, $f_j(x_i) = x_i$ and x_i is a fixed point of f_j. But f_j has a unique fixed point x_j. Hence $x_i = x_j$. Thus $x_1 = x_2 = ... = x_i = ... = x$, is a common unique fixed point for all f_i $(i = 1, 2, ...)$.

We now prove a fixed point theorem of the alternative which is a generalization of a theorem of DIAZ and MARGOLIS [3]. The BANACH contraction theorem and a theorem due to CHU and DIAZ [2] will be easy corrollaries to our theorem.

Theorem 3. *Let X be a generalized complete metric space and K: $X \to X$ any mapping with a right inverse, i.e. $KK^{-1} = 1$, the identity mapping. Let f: $X \to X$ be any mapping. Suppose $g = K^{-1} f K$ is a contraction in the sense that is satisfies the following condition:*

(a) *There exists a constant k with $0 < k < 1$ such that whenever $d(x, y) < \infty$, then $d(gx, gy) < k\, d(x, y)$.*

Let $x_0 \in X$, *then the following alternative holds: either*

(A) *for every integer* $r = 0, 1, 2, \ldots$ *one has* $d(g^r x_0, g^{r+1} x_0) = \infty$,

or

(B) f *has a fixed point in* X.

Proof. Consider the sequence of numbers

$$d(x, gx_0),\ d(gx_0, g^2 x_0),\ \ldots,\ d(g^r x_0, g^{r+1} x_0),\ \ldots .$$

There are two mutually exclusive possibilities, either for every integer $r = 0, 1, 2, \ldots$, $d(g^r x_0, g^{r+1} x_0) = \infty$, which is precisely alternative (A); or (B′) for some integer $r = 0, 1, 2, \ldots, d(g^r x_0, g^{r+1} x_0) < \infty$.

It now remains to show that (B′) implies (B). Suppose (B′) holds. Let $N = N(x_0)$ denote a particular integer of the set of integers $r = 0, 1, 2, \ldots$ such that $d(g^r x_0, g^{r+1} x_0) < \infty$. Then by (a) since $g\, d(g^N x_0, g^{N+1} x_0) < \infty$, it follows that

$$d(g^{N+1} x_0, g^{N+2} x_0) = d(g\, g^N x_0, g\, g^{N+1} x_0) \leqslant k\, d(g^N x_0, g^{N+1} x_0) < \infty .$$

By induction it can be shown that

$$d(g^{N+1} x_0, g^{N+r+1} r x_0) \leqslant k^r d(g^N x_0, g^{N+1} x_0) < \infty \quad \text{for all } r = 0, 1, 2, \ldots .$$

In other words, for any integer $n > N$,

$$d(g^N x_0, g^{N+1} x_0) \leqslant k^{n-N} d(g^N x_0, g^{N+1} x_0) < \infty .$$

Using the triangle inequality it follows that, for $n > N$,

$$d(g^N x_0, g^{n+r} x_0) \leqslant \sum_{i=1}^{r} d(g^{n+i-1} x_0,\ g^{n+i} x_0)$$

$$\leqslant \sum_{i=1}^{r} k^{n+i-1-N} d(g^N x_0, g^{N+1} x_0) \leqslant k^{n-N}(1 - k^r)/(1 - k) \cdot d(g^N x_0, g^{N+1} x_0) ,$$

where $r = 1, 2, \ldots$.

Since $0 < k < 1$, the sequence of successive approximations $x_0, x_1, \ldots,$ $g^n x_0, \ldots$ is a d-CAUCHY sequence, and since X is a generalized complete metric space, is d-convergent, i.e. $\lim_{n\to\infty} d(g^n x_0, x) = 0$ for some $x \in X$. We now show that x is a fixed point of g. Whenever $n > N$ it follows from (a) and the triangular inequality that

$$0 \leqslant d(x, gx) \leqslant d(x, g^n x_0) + d(g^n x_0, gx) \leqslant d(x, g^n x_0) + k\, d(g^{n-1} x_0, x) .$$

Taking the limit as $n \to \infty$ it follows that $d(x, gx) = 0$. Thus $g(x) = x$ and x is a fixed point of g. Hence $K^{-1}fK(x) = x$ and $KK^{-1}fK(x) = Kx$. This implies that $f(Kx) = Kx$. So Kx is a fixed point of f.

Remarks.

1) In case $K^{-1}fK$ is replaced by a contraction map f, then we get a known result due to DIAZ and MARGOLIS [3].

2) In case X is a complete metric space, then alternative (A) is excluded and hence $K^{-1}fK$ has a fixed point in X which is obviously unique. This theorem has been given by CHU and DIAZ [1].

3) In case X is a complete metric space and $K^{-1}fK$ is replaced by a contraction map f then we get a well-known theorem of BANACH, which states that a contraction map on a complete metric space has a unique fixed point.

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