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W. MIELANTS (*)

The Enumeration of Classes of Homotopic Equivalent Topologies. (**)

1. - Introduction.

1.1. – Consider a finite topological space $F \equiv (X_n, \tau)$ where n denotes the cardinality of the set X_n. If $M \subseteq X_n$, the open hull $Q(M)$ is the intersection of all open subsets of F containing M. Of course Q satisfies the axioms of the KURATOWSKI closure operator. We write $Q(x)$ for the open hull $Q(\{x\})$ of a point $x \in X_n$. Obviously the system $\{\emptyset, Q(x)//x \in X_n\}$ is a basis of F which refines every basis. Now we define a relation $\leqslant$ on X_n by saying $x \leqslant y \Leftrightarrow$ $\Leftrightarrow x \in Q(y)$ (equivalently $Q(x) \subseteq Q(y)$). Clearly $\leqslant$ is reflexive and transitive. It is easy to see that the map f: $F_1 \to F_2$ is continuous if and only if f is isotonic.

A finite topological space is T_0 if and only if $\leqslant$ is a partial order, and totally disconnected if and only if $\leqslant$ is an equivalence relation (in this case the closure operator Q satisfies the MACLANE-STEINITZ exchange property).

The proces is reversible. Let $(X_n, \leqslant)$ be a finite set with a quasi-ordering. For each $x \in X_n$ let $Q(x) = \{y \in X_n // y \leqslant x\}$. Then the system $\{\emptyset, Q(x)//x \in X_n\}$ is a basis of a topology on X_n. Hence it is easy to prove that there exist a bijective covariant functor from the category of the finite quasi-orderings (respectively: partial orderings, equivalence relations) with the isotonic functions as morphisms, to the category of finite topological spaces (respectively: T_0-spaces, totally disconnected spaces) with the continuous functions as morphisms.

(*) Indirizzo: Riinglaan 102, 2610 Wibrijk, Belgio.
(**) Ricevuto: 9-XII-1971.

1.2. – It is easy to see that every finite topological space is homotopic equivalent with a finite T_0-topological space. Indeed the identification-topology defined by the equivalence-relation: two points are equivalent if and only if they have all the same open neighbourhoods is a T_0-topological space, which is a strong deformation-retract of this topological space [**22**].

Hence the number of classes of homotopic equivalent T_0-topologies on a finite set will be the same as for arbitrary topologies. We denote this number by $M(n)$.

2.1. – Definition. A point x of an arbitrary finite T_0-topological space is called *linear* if there exist in the associated ordering $P_\leqslant$ a point $y<x$ such that $\forall z \in P_\leqslant$: $z<x \Rightarrow z \leqslant y$.

2.2. – Definition. A point x of an arbitrary finite T_0-topological space is called *colinear* if there exist in the associated ordering $P_\leqslant$ a point $y>x$ such that $\forall z \in P_\leqslant$: $z>x \Rightarrow z \geqslant y$.

2.3. – Definition. The *core*: $C(F)$ of an arbitrary finite topological space F is a subspace $C(F)$ of F which is a strong deformation-retract of F, which is a T_0-space, and which has no linear and colinear points.

It has been proved by R. E. Stong [**22**] that every finite space has a core; that all cores of a finite space are homeomorphic equivalent and that two finite spaces F_1 and F_2 are homotopic equivalent if and only if the core of F_1 is homeomorphic equivalent with the core of F_2 or $F_1 \simeq F_2 \Leftrightarrow C(F_1) \cong C(F_2)$.

2.4. – Remark. If F is contractible then $C(F)$ will be the trivial topology on one point.

2.5. – Theorem. *If we denote the number of classes of homeomorphic equivalent T_0-topologies without linear and colinear points on a finite set with cardinality n by $C(n)$ then we will have*

$$M(n) = \sum_{m=1}^{n} C(m) .$$

Indeed: with each class of homotopic equivalent topologies on n points there corresponds exactly one class of homeomorphic equivalent T_0-topologies without linear and colinear points (the common core of these spaces) on $m \leqslant n$ points. Conversely with each class of homeomorphic T_0-spaces without linear and colinear points on m points corresponds exactly one class of homotopic equivalent topologies on $n \geqslant m$ points of which it is the common core.

2.6. – So the problem of the enumeration of classes of homotopic equivalent topologies on a finite set is reduced to the enumeration of classes of homeomorphic equivalent T_0-topologies without linear and colinear points, or equivalently to the enumeration of ordering-structures without linear and colinear points.

The problem of the enumeration of ordering-structures without linear and colinear points is of course very difficult since it is an enumeration problem concerning classes of isomorphic structures which are locally restricted (indeed no point of the ordering may cover exactly one point and no point may be covered by exactly one point). Even in the most simple case of rank 1 (where the great difficulty of the transitivity-property falls off) the problem is reduced to the enumeration of classes of isomorphic bichromatic graphs (directed, and without loops and multiple edges) where the valency of every vertice is different from 1. The vertices of the first colour are points which covers other points, vertices of the second colour are points which are covered by other points, and isolated vertices have an arbitrary colour. Because no point of the ordering covers exactly one point or is covered by exactly one point the valency of every vertice in the corresponding graph had to be different from 1. Using the most powerful methods of combinatorial theory R. C. READ has solved this problem for the total number [**15**] and remarked that the enumeration of classes of isomorphic different such graphs is of an higher order of difficulty.

3. – *The enumeration of classes of homotopic equivalent topologies on X_n with $n \leqslant 7$ and the calculation of the singular homology groups of each class.*

3.1. – With each T_0-space without linear and colinear points on a set with less than eight points, I shall associate a polyheder whose homotopy groups, singular homology and cohomology groups are isomorphic with those of each topological space whose core is homeomorphic with this space.

I shall use the following results of M. C. MCCORD [**4**].

With each finite T_0-topological space F we can associate the abstract simplicial complex $\Phi(F)$ whose simplexes are the well-orderings in the associated ordering of F. Φ is a covariant functor from the category of finite T_0-topologies with the continuous functions as morphisms to the category of the simplicial complexes with the simplicial transformations as morphisms. We denote by $|\Phi(F)|$ the polyheder of a geometrical realisation of $\Phi(F)$ in $\mathfrak{R}^m$ (m sufficiently great). With each point p of $|\Phi(F)|$ with carier $(u_0, u_1, \ldots, u_t)$ with $u_0 < u_1 < < \ldots < u_t$ (each simplex is a well ordering we define $f(p) = u_0$. M. C. CORD [**4**] has proved that $f\colon |\Phi(F)| \to F$ is a weak homotopy-equivalence or a continuous function which induces maps

$$f_*\colon \quad \pi_i(|\Phi(F)|, x) \to \pi_i(F, f(x))$$

which are isomorphisms for all $x \in X_n$ and all $i \geqslant 0$ [for $i = 0$ « isomorphism » is understood to mean simply « 1-1-correspondence »]. It is a well known theorem of J. H. C. WHITEHEAD that every weak homotopy equivalence induces isomorphisms on singular homotopy-groups (hence also on singular cohomologyrings).

So with each core F on less than eigth points, we can construct $|\Phi(F)|$ and calculate the homology-groups of these polyhedra.

3.2. – For $n \leqslant 7$ we have:

n	1	2	3	4	5	6	7
$C(n)$	1	1	1	2	4	11	32
$M(n)$	1	2	3	5	9	20	52

We use the additive group of the integers Z as coefficient-group and we denote the free Abelian group on m generators by

$$A_m = Z + Z + \dots + Z \ (m \text{ terms}) .$$

The 52 classes of homotopic equivalent topological spaces with cardinality less than 8 consist of

1) *32 classes of connected topologies* $[H_0 = Z]$

— The class of contractible topologies
— 2 classes with $H_1 = Z$ $H_i = 0$ $i \geqslant 2$
— 5 classes with $H_1 = A_2$ $H_i = 0$ $i \geqslant 2$
— 9 classes with $H_1 = A_3$ $H_i = 0$ $i \geqslant 2$
— 7 classes with $H_1 = A_4$ $H_i = 0$ $i \geqslant 2$
— 2 classes with $H_1 = A_5$ $H_i = 0$ $i \geqslant 2$
— 2 classes with $H_1 = A_6$ $H_i = 0$ $i \geqslant 2$
— 1 class with $H_1 = 0$ $H_2 = Z$ $H_i = 0$ $i \geqslant 3$
— 3 classes with $H_1 = 0$ $H_2 = A_2$ $H_i = 0$ $i \geqslant 3$.

2) *11 classes of topologies with* 2 *components* $[H_0 = A_2]$

— 1 class with	$H_i = 0$	$i \geqslant 1$		
— 2 classes with	$H_1 = Z$	$H_i = 0$	$i \geqslant 2$	
— 5 classes with	$H_1 = A_2$	$H_i = 0$	$i \geqslant 2$	
— 1 class with	$H_1 = A_3$	$H_i = 0$	$i \geqslant 2$	
— 1 class with	$H_1 = A_4$	$H_i = 0$	$i \geqslant 2$	
— 1 class with	$H_1 = 0$	$H_2 = Z$	$H_i = 0$	$i \geqslant 3$.

3) *4 classes of topologies with* 3 *components* $[H_0 = A_3]$

— 1 class with	$H_i = 0$	$i \geqslant 1$	
— 1 class with	$H_1 = Z$	$H_i = 0$	$i \geqslant 2$
— 2 classes with	$H_1 = A_2$	$H_i = 0$	$i \geqslant 2$.

4) *2 classes of topologies with* 4 *components* $[H_0 = A_4]$

— 1 class with	$H_i = 0$	$i \geqslant 1$	
— 1 class with	$H_1 = Z$	$H_i = 0$	$i \geqslant 2$.

5) 1 class with respectively 5, 6 and 7 components all homotopic equivalent with the discrete topology on respectively 5, 6 and 7 points.

4. - Existence-problems.

4.1. – Using the results of the work of McCord it is easy to prove that for every finitely generated group there exist a finite connected topological space having this group as fundamental group of Poincaré. So we can ask the following existence-problem: given an arbitrary finitely generated group, what are the necessary and sufficient conditions to which the cardinality of a set had to satisfy such that a connected topology can be defined on it having this group as fundamental group.

4.2. – For $\pi_1 = Z$ it is $n \geqslant 4$. The topological space with minimal cardinality beeing connected and having this group as fundamental group is the following topology on 4 points $\{s_1, s_2, s_3, s_4\}$ with basis: $\{\emptyset; \{s_1\}; \{s_2\}; \{s_1, s_2, s_3\}; \{s_1, s_2, s_4\}\}$.

It is the connected topological space with minimal cardinality beeing non-contractible, and there exist a week homotopy equivalence from the circle to this space.

4.3. – For $\pi_1 = F_2$ (the free group with 2 generators) it is $n \geqslant 5$. The topological spaces with minimal cardinality beeing connected and having this group as fundamental group are the following two (homeomorphic different) topologies on 5 points $\{s_1, s_2, s_3, s_4, s_5\}$ with basis respectively: $\{\emptyset; \{s_1\}; \{s_2\}; \{s_1, s_2, s_3\}; \{s_1, s_2, s_4\}; \{s_1, s_2, s_5\}\}$ and $\{\emptyset; \{s_1\}; \{s_2\}; \{s_3\}; \{s_1, s_2, s_3, s_4\}; \{s_1, s_2, s_3, s_5\}\}$.

4.4. – More generally for $\pi_1 = F_m$ it is $n \geqslant k$ where $k = \min(n_1; n_2)$ where n_1 is the least even number such that $\frac{1}{4}(n_1 - 2)^2 \geqslant m$ and n_2 the least odd number such that $\frac{1}{4}(n_2 - 2)^2 + \frac{1}{4} \geqslant m$. If $k = n_1$, the connected topological space with minimal cardinality having this group as fundamental group is the following topology on $n_1 = 2n_1'$ points s_i $(i = 1, 2, \ldots, n_1)$ with basis $\{\emptyset; \{s_1\}; \ldots; \{s_{n_1'}\}; \{s_1, s_2, \ldots, s_{n_1'}, s_{n_1'+1}\}; \ldots \{s_1, s_2, \ldots s_{n_1'}, s_{n_1}\}\}$.

If $k = n_2$ it are the following 2 (homeomorphic different) topologies on $n_2 = 2n_2' + 1$ points s_i $(i = 1, 2, \ldots, n_2)$ with basis respectively $\{\emptyset; \{s_1\}; \{s_2\}, \ldots; \{s_{n_2'}\}; \{s_1, s_2, \ldots, s_{n_2'}, s_{n_2'+1}\}; \ldots \{s_1, s_2, \ldots, s_{n_2'}, s_{n_2}\}\}$ and $\{\emptyset; \{s_1\}; \{s_2\}; \ldots, \{s_{n_2'}\}; \{s_{n_2'+1}\}; \{s_1, s_2, \ldots s_{n_2'+1}, s_{n_2'+2}\} \ldots \{s_1, s_2, \ldots s'_{n_2'+1}, s_{n_2}\}\}$.

4.5. – For $\pi_1 = Z_2$: $n \geqslant 31$ is a sufficient condition. Indeed on a set with 31 points we can construct a connected topology whose fundamental group is Z_2 and whose points can be considered as the vertices of the first barycentric division of the minimal dissection of the real projective plane; and where the minimal open set $Q(x)$ of a point x is defined to be the set of vertices of the simplex in this dissection whose barycenter is the point x. (By a theorem of McCord [4] we can then prove that there exist a weak homotopy equivalence from the real projective plane to this space and so $\pi_1 = Z_2$).

4.6. – More generally for $\pi_1 = Z_p$: $n \geqslant f(p)$ is a sufficient condition where $f(p)$ denotes the number of vertices of the first barycentric division of the minimal dissection of the Lenz-space $L(p, 1)$.

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Summary.

The enumeration problem concerning the number of classes of homotopic equivalent topologies on a finite set is reduced to the enumeration of certain locally restricted ordering-structures.

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