Riv. Mat. Univ. Parma (3) 1 (1972), 83-89

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On the Number of Ordering-Structures. (**)

1. - Introduction.

1.1. – The structure of a relation, as understood for instance by RUSSELL [**19**] and CARNAP [**3**], is simply the class of all relations to which it is isomorphic (what WHITEHEAD and RUSSELL at first called the relation-number [**26**]).

An important unsolved enumeration problem ([**5**], [**8**], [**10**], [**21**]) is the determination of the number of ordering-structures on a finite set. It is well known ([**1**], [**2**], [**4**], [**7**], [**21**]) that there exist a bijective covariant functor from the category of finite orderings with the isotonic functions as morphisms to the category of finite T_0-topologies with the continuous functions as morphisms.

Hence, many papers concerning this enumeration problem use a topological terminology. The object of this paper is to find a reasonable lower bound for the number of ordering-structures on a finite set with cardinality n, which I shall denote by $N(n)$.

1.2. – I shall start with results obtained by D. A. KLARNER [**10**]. A poset is called graded if at least one rank function can be defined on it. A rank function is a map from the points of the poset to the chain of integers such that if x covers y then

$$f(x) = f(y) + 1 .$$

The rank of a poset is the greatest length of a chain in it. If we denote the number of classes of isomorphic graded orderings with rank h on a finite set with cardinality n by $N_g(h, n)$; then D. A. KLARNER obtained the following result.

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Consider an arbitrary decomposition of the number n into exactly h non negative parts:

$$(v_1, v_2, \ldots, v_h)\ ; \qquad v_i \geqslant 0\ ; \qquad \sum_{i=1}^{h} v_i = n\ ; \qquad v_i \in Z .$$

With each permutation $\bar{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ of the permutation-group: $S_{(v_1, v_2, \ldots, v_h)} = \prod_{\otimes}^{i=1\to h} \mathrm{sym}(v_i)$, where $\mathrm{sym}(v_i)$ denotes the symmetrical group of degree v_i, we associate the following number:

$$C(\bar{\pi}) = \sum_{j=1}^{k} \sum_{p=1}^{v_j} \sum_{q=1}^{v_{j+1}} (p, q)\ C(\pi_j, p)\, C(\pi_{j+1}, q)\ ,$$

where (p, q) denotes the greatest common divisor of p and q, and $c(\sigma, j)$ denotes the number of cycles of length j of the permutation σ.

Consider then the following number

$$p_h(n) = \sum_{(v_1 \ldots v_h)} \left[\frac{1}{\prod_{i=1}^{h} v_i\,!} \sum_{\bar{\pi} \in S_{(v_1 \ldots v_h)}} 2^{c(\bar{\pi})} \right] ,$$

when the sum extends on all possible compositions $(v_1 \ldots v_h)$ of n into exactly h non-negative parts. Consider then the following generating function

$$P_h(x) = \sum_{n=0}^{\infty} p_h(n)\, x^n .$$

Consider also the functions $B_h(x) = P_h(x)/P_{h-1}(x)$.

If $B_h(x) = \sum_{n=0}^{\infty} b_h(n)\, x^n$, then $N_g(h, n) = b_{h+1}(n) - b_h(n)$.

We shall use these numbers $N_g(h, n)$ obtained by D. A. KLARNER to find a reasonable lower bound of the number of ordering-stuctures on a finite set.

1.3. – Remark. H. SHARP [**21**] obtained the following results for $N(n)$ and $N_c(n)$ for $n \leqslant 5$. (By $N_c(n)$ we denote the number of classes of isomorphic *connected* orderings):

n	1	2	3	4	5
$N_c(n)$	1	1	3	10	44
$N(n)$	1	2	5	16	63

2. - A lower bound of the number of ordering-structures.

2.1. – Lemma 1. *By adding one point to a graded poset with rank h, ($h \geqslant 3$), we can construct $\varphi(h) = \sum_{a=0}^{h-3} 2^a(2^{h-a-2} - 1)$ classes of isomorphic non-graded posets with rank h.*

Proof. Consider a graded ordering $P_\leqslant$ with rank h on a finite set X_n with cardinality n.

There exists at least one chain $C_\leqslant$ in $P_\leqslant$ with length h. The minimal element of this chain will be denoted by c_0; and the element of $C_\leqslant$ which covers c_i, will be denoted by c_{i+1} ($i = 0, 1, 2, \ldots, h$). Consider now a new point p and the following orderings defined on $X_n \cup \{p\}$. We let the point p cover the points $(c_{i_1}, c_{i_2}, \ldots, c_{i_m})$ with $0 \leqslant i_1 < i_2 \ldots < i_m \leqslant h - 3$, and we let the points $(c_{j_1}, c_{j_2}, \ldots, c_{j_t})$ with $i_m + 3 \leqslant j_1 < j_2 \ldots < j_t \leqslant h$ cover the point p. In this way we obtain non-graded posets on $n+1$ points with rank h. The poset is non-graded because if there exist a rank-function f on this new poset (we can always assume that $f(c_0) = 0$), then $f(p) = i_m + 1$ and so $f(c_{j_1}) = i_m + 2$ and this in contradiction with the fact that $j_1 \geqslant i_m + 3$.

Consequently there is no rankfunction on this new poset and hence it is non-graded. The rank is always h because the lenght of an arbitrary chain passing trough p is less than h. Now we prove that the non-graded poset obtained by this construction is then and only then iscmorphic with the non-graded poset obtained by the same construction starting with the points

$$(c_{i'_1}, c_{i'_2}, \ldots, c_{i'_{m'}}) \qquad 0 \leqslant i'_1 < i'_2 \ldots < i'_{m'} \leqslant h - 3$$

and

$$(c_{j'_1}, c_{j'_2}, \ldots, c_{j'_{t'}}) \qquad i'_m + 3 \leqslant j'_1 < j'_2 \ldots < j'_{t'} \leqslant h$$

when

$$(c_{i_1} c_{i_2} \ldots c_{i_m}) \equiv (c_{i'_1} c_{i'_2} \ldots c_{i'_{m'}}) \quad \text{and} \quad (c_{j_1} c_{j_2} \ldots c_{j_t}) \equiv (c_{j'_1} c_{j'_2} \ldots c_{j'_t}) .$$

Indeed if we denote the first ordering by $T_\leqslant$ and the second by $T'_{\leqslant'}$ and if we assume that or $(c_{i_1} \ldots c_{i_m}) \not\equiv (c_{i'_1} \ldots c_{i'_{m'}})$ or $(c_{j_1} \ldots c_{j_t}) \not\equiv (c_{j'_1} \ldots c_{j'_t})$, and that there exists an isomorphism Φ: $T_\leqslant \to T'_{\leqslant'}$ then there are 2 possibilities:

a) $$\Phi(p) = p .$$

Then Φ will induce an automorphism of $P_{\leqslant}$. But $\Phi(c_i) = c_i$ $(i = 0, 1, 2, ..., h)$ because c_i is the only point k with $f(k) = h$ which covers, or is covered by the point p. So $(c_{i_1} ... c_{i_m}) \equiv (c_{i'_1} ... c_{i'_{m'}})$ and $(c_{j_1} ... c_{j_t}) \equiv (c_{j'_1} ... c_{j'_{t'}})$ which is a contradiction.

b) $$\Phi(p) \neq p .$$

If $A_{\leqslant}$ is an arbitrary poset, we shall denote by $S(A_{\leqslant})$ the set of all points x such that there exist a chain of length $k \geqslant 3$ in $A_{\leqslant}$: $(a_i, a_{i+1}, ..., a_{i+k})$ where a_{i+k} covers x and x covers a_i. Of course if $A_{\leqslant}$ is a graded poset then $S(A_{\leqslant}) = \emptyset$. In this case we have by construction $S(T_{\leqslant}) = S(T'_{\leqslant'}) = \{p\}$.

Now it is evident that if Φ is an isomorphism of $T_{\leqslant} \to T'_{\leqslant'}$ that $\Phi_*[S(T_{\leqslant})] = S(T'_{\leqslant'})$ so $\Phi(p) = p$ which is a contradiction.

Now we prove that there exist exactly $\varphi(h) = \sum_{a=0}^{h-3} 2^a(2^{h-a-2} - 1)$ possible different choices of $(c_{i_1} ... c_{i_m})$ and $(c_{j_1} ... c_{j_t})$ with $0 \leqslant i_1 < i_2 ... < i_m \leqslant h - 3$ and

$$i_m + 3 \leqslant j_1 < j_2 ... \leqslant h .$$

Indeed for a given value $i_m = a$ there are 2^a possibilities for $(c_{i_1} ... c_{i_m})$: all subsets of $\{c_0, c_1, ..., c_{a-1}\}$ also the empty subset; and $(2^{h-a-2} - 1)$ possibilities for $(c_{j_1}, c_{j_2}, ..., c_{j_t})$; all non-empty subsets of $\{c_{a+3}, c_{a+4}, ..., c_h\}$.

2.2. – Lemma 2. *By adding k points $(p_1, p_2, ..., p_k)$ to a graded poset with rank h, we can construct $\binom{\varphi(h) + h - 1}{k}$ classes of isomorphic non-graded posets with rank h $(h \geqslant 3)$.*

Proof. By adding one point we can construct $\varphi(h)$ classes of isomorphic non-graded posets with rank h. We shall denote these constructions by T_1, T_2, ..., $T_{\varphi(h)}$. For each of the points $(p_1, p_2, ..., p_h)$ we start with an arbitrary of these constructions. If we denote the construction we use by adding the point p_r $(r = 1, 2, ..., k)$ by T_{i_r}, then we shall denote the total construction by $K = (T_{i_1}, T_{i_2}, ..., T_{i_k})$ $(0 \leqslant i_r \leqslant \varphi(h))$. It is not necessary that these constructions are all different, they may even be all the same (by using the same construction for each point p_r). So there are exactly $\binom{\varphi(h) + k - 1}{k}$ different possible constructions $K = (T_{i_1}, T_{i_2}, ..., T_{i_r})$. We prove now that 2 non-graded posets $T_{\leqslant}$ and $T'_{\leqslant'}$ obtained by adding k points $p_1 ... p_h$ to a graded poset with rank h using respectively the constructions $K = (T_{i_1} ... T_{i_h})$ and $K' = (T_{i'_1} ... T_{i'_h})$ are then and only then isomorphic when $K \equiv K$!. Indeed assume that $K \not\equiv K'$ then there exist at least one construction $T_{i_r} \in K$ such that $T_{i_r} \notin K'$.

By construction we have $S(T_{\leqslant}) = S(T'_{\leqslant'}) = \{p_1 p_2 \dots p_k\}$. If there should exist an isomorphism Φ: $T_{\leqslant} \to T'_{\leqslant'}$, then $\Phi_*(S(T_{\leqslant})) = S(T'_{\leqslant'}) = \{p_1 p_2 \dots p_h\}$. But it $\Phi(p_r) = p_m$ then we must have $T_{i'_m} \equiv T_{i_r}$ which is in contradiction with the fact that $T_{i_r} \notin K'$.

2.3. – Theorem. *There exist at least*

$$\boxed{D(h, n) = \sum_{m=h+1}^{n} \binom{\varphi(h) + n - m - 1}{n - m} N_g(h, m)}$$

ordering-structures with rank h on a finite set with cardinality n $(h \geqslant 3)$.

Proof. Consider the $N_g(h, m)$ graded posets with rank h on a set with cardinality m $(h + 1 \leqslant m < n)$. By adding $n - m$ points $(p_1 p_2 \dots p_{n-m})$ we can construct [see Lemma 2] $\binom{\varphi(h) - n - m - 1}{n - m}$ $N_g(h, m)$ non-graded posets with rank h.

We prove now that no two of them are isomorphic. Consider two non-isomorphic graded posets with rank h on m points $P_{\leqslant}$ and $P'_{\leqslant'}$, and consider the non-graded posets $T_{\leqslant}$ and $T_{\leqslant'}$ obtained by adding the points $p_1 p_2 \dots p_{n-m}$ respectively to $P_{\leqslant}$ and $P'_{\leqslant'}$ using arbitrary constructions.

Now it is easy to see that there can't exist an isomorphism Φ of $T_{\leqslant} \to T'_{\leqslant'}$: Indeed, since by construction $S(T_{\leqslant}) = S(T'_{\leqslant'}) = \{p_1 p_2 \dots p_{n-m}\}$ and since $\Phi_* S(T_{\leqslant}) = S(T'_{\leqslant'})$ Φ must induce an isomorphism between $P_{\leqslant}$ and $P'_{\leqslant'}$, which is in contradiction with the fact that $P_{\leqslant}$ is non-isomorphic with $P'_{\leqslant'}$. Now, consider an arbitrary graded poset with rank h on m points: $P^1_{\leqslant}$. By adding $n - m$ points we can construct $\binom{\varphi(h) + n - m - 1}{n - m}$ classes of isomorphic non-graded posets on n points. Consider next a graded poset with rank h on m points $P^2_{\leqslant}$ non-isomorphic with $P^1_{\leqslant}$. By adding $n - m$ points we can construct the same number of classes of isomorphic graded posets with rank h on n points, and we have proved that these classes are all different from those of the first construction. Because there exist exactly $N_g(h, m)$ classes of isomorphic graded posets with rank h on m points, we can construct $N_g(h, m) \cdot \binom{\varphi(h) + n - m - 1}{n - m}$ classes of isomorphic non-graded posets with rank h on n points.

Consider also the $N_g(h, m')$ classes of isomorphic graded posets with rank h on $m' \neq m$ points. By adding $n - m'$ points we can construct $N_g(h, m') \cdot \binom{\varphi(h) + n - m' - 1}{n - m'}$ classes of isomorphic non-graded posets with rank h on m' points which are all different from the classes obtained by starting with a graded poset on $m \neq m'$ points.

Indeed if $T_{\leqslant}$ and $T'_{\leqslant'}$ are two arbitrary non-graded posets with rank h obtained by adding respectively $n-m$ and $n-m'$ points to a graded poset with rank h on respectivily m and m' points, and if there should exist an isomorphism.

$$\Phi\colon\quad T_{\leqslant}\to T'_{\leqslant'}\,,\qquad \text{then}\qquad \Phi_*[S(T_{\leqslant})]=S(T'_{\leqslant'})\qquad \text{but}\qquad S(T_{\leqslant})=S(T'_{\leqslant'})$$

which is in contradiction with the fact that $m\neq m'$. However $N_g(h, m) = 0$ for $m \leqslant h$ and hence we can construct

$$D(h, n)=\sum_{m=h+1}^{n}\binom{\varphi(h)+n-m-1}{n-m}N_g(h, m)$$

classes of isomorphic posets with rank h on a finite set with cardinality n.

2.4. – Remark 1. $D(h, n)$ is also a lower bound of the number of classes of homeomorphic T_0-topologies with small inductive dimension h on a finite set with cardinality n.

If we define the chromatic number $X(F)$ of a finite topological space F by the minimal number of colours which is necessary to colour all points of the space in such a way that two points have different colours when one of them lies in all the open neighbourhouds of the other, then $D(h, n)$ will also be a lower bound of the number of classes of homeomorphic T_0-topologies with chromatic number $h+1$ on n points.

2.5. – Remark 2. For $h=0, 1, 2$ $D(h, n)=N_g(h, n)=N(h, n)$ since there exist no non-graded posets with renk $h<3$.

2.6. – Corollary. *We obtained the following lower bound for the number of ordering-structures on a finite set with cardinality n*

$$N(n)\geqslant\sum_{h=0}^{n-1}\sum_{m=h+1}^{n}\binom{\varphi(h)+n-m-1}{n-m}N_g(h, m)\,.$$

Bibliography.

[1] P. Alexandroff, *Discrete Räume,* Math. Sb. (N. S.) **2** (1937), 501-518.

[2] D. W. Bass, *Dimension and finite spaces,* J. London Soc. **44** (1969), 159-162.

[3] R. Carnap, *Logical Foundations of Probability,* Chicago 1950.

[4] M. C. McCord, *Singular homology groups and homotopy-groups of finite topological spaces*, Duke Math. J. **33** (1966), 465-474.

[5] R. L. Davis, *The number of structures of finite relations*, Proc. Amer. Math. Soc. **4** (1953), 486-495.

[6] N. G. DeBruijn, *Polya's theory of counting*, Ch. 5 of « Applied Combinatorial Mathematics » (E. G. Beckenback; ed.), Wiley, New York 1964, 144-184.

[7] F. Harary, J. W. Evans and M. S. Lynn, *On the computer-enumeration of finite topologies*, Comm. A.C.M. **10** (1967), 295-298.

[8] F. Harary, *Enumeration under group action: unsolved graphical enumeration problems*, IV, J. Combinat. Theory **8** (1970), 1-11.

[9] A. D. Klarner, *The number of graded partially ordered sets*, J. Combinat. Theory **6** (1969), 12-19.

[10] A. D. Klarner, *The number of classes of isomorphic graded partially ordered sets*, J. Combinat. Theory **9** (1970), 412-419.

[11] D. Kleitman and B. Rothshild, *The number of finite topologies*, Trans. Amer. Math. Soc. **25** (1970), 276-282.

[12] V. Krishnamurthy, *On the number of topologies on a finite set*, Amer. Math. Monthly **73** (1966), 154-157.

[13] W. Mielants, *Autohomeomorphism groups of finite topological spaces*, (to appear).

[14] W. Mielants, *On the number of topologies with given small inductive dimension*, (to appear).

[15] G. Polya, *Kombinatorische Anzahlbestimmungen fur Gruppen, Graphen und chemische Verbindungen*, Acta Math. **68** (1937), 145-254.

[16] R. C. Read, *The enumeration of locally restricted graphs*, (I), (II); (I) J. London Math. Soc. **34** (1959), 417-436; (II) J. London Math. Soc. **35** (1960), 344-351.

[17] R. C. Read, *Some applications of a Theorem of DeBruijn Graph Theory and Theoretical Physics*, Academic Press, New York 1967.

[18] G. C. Rota, *On the foundations of combinatorial theory; theory of Möbius functions*, Z. Wahrscheinlichkeitstheory and Verw. Gebiete **2** (1964), 340-368.

[19] B. Russell, *Introduction to Mathematical Philosophy*, London 1920.

[20] A. Shafaat, *On the number of topologies definable on a set*, J. Austral. Math. Soc. **00** (1966), 194-198.

[21] H. Sharp, *Quasi-orderings and topologies on finite sets*, Proc. Amer. Math. Soc. **17** (1966), 1344-1349.

[22] H. Sharp, *Cardinality of finite topologies*, J. Combinat. Theory **5** (1968), 82-86.

[23] D. Stephen, *Topology on finite sets*, Amer. Math. Monthly **75** (1968), 734-741.

[24] R. Stong, *Finite topological spaces*, Trans. Amer. Math. Soc. **123** (1966), 325-340.

[25] A. Tarski, *On the calculus of relations*, J. Symbolic Logic **6** (1941), 73-89.

[26] A. N. Whitehead and B. Russell, *Principia Mathematica*, Vol. I, Cambridge, University Press 1910.

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