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On Common Fixed Points. (**)

1. - Introduction.

Given a mapping $T: X \to X$ (a set), a point $x_0 \in X$ is said to be a fixed point of T if $Tx_0 = x_0$. The well known BANACH's Contraction Theorem states: If (X, d) is a complete metric space and $T: X \to X$ is a contraction mapping, i.e. a mapping for which there exists a real number k, $0 \leqslant k < 1$ such that

$$d(Tx, Ty) \leqslant k\, d(x, y) \quad \text{for all} \quad x, y \in X\,,$$

then T has a unique fixed point or equivalently the equation $Tx = x$ has a unique solution.

This theorem has been extensively used in proving the existence and uniqueness of solutions to various differential and integral equations.

It has been of interest in Analysis to study the existence of the common fixed point of two or more mappings defined over the same space. DE MARR [2], [3], KANNAN [4] and others have worked in this direction.

The result of KANNAN states:

Theorem 1.1. *Let (X, d) be a comple metric space. If T_1 and T_2 are two mappings of X into itself satisfying,*

$$\text{(A)} \qquad d(T_1 x, T_2 y) \leqslant \alpha\{d(x, T_1 x) + d(y, T_2 y)\}, \quad \textit{for all} \quad x, y \in X$$

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where α is a real number such that $0 \leqslant \alpha < \frac{1}{2}$, then T_1 and T_2 have a unique common fixed point.

The main aim of this paper is to generalize the Theorem of KANNAN and to give one further generalization.

2. – Remark 2.1. If in the above theorem, the mappings T_1 and T_2 fail to satisfy the condition (A), but however they satisfy a weaker condition (B), as given below then the conclusion of the theorem still holds.

Thus we prove the following.

Theorem 2.1. *Let T_1 and T_2 be two mappings of a complete metric space (X, d) into itself. If there exist two non-negative real numbers α and β such that $\alpha + \beta < 1$ and,*

$$\text{(B)} \qquad d(T_1 x, T_2 y) \leqslant \alpha\, d(x, T_1 x) + \beta\, d(y, T_2 y) \qquad \text{for all} \quad x, y \in X\,,$$

then T_1 and T_2 have a common unique fixed point.

Proof. First we construct a sequence of alternate iterates of the two mappings in the usual way and prove it to be a CAUCHY sequence.

Let x_0 be an arbitrary point in X. Set a sequence $\{x_n\}_{n=1}^{\infty}$ of points in X as $x_1 = T_1 x_0$, $x_2 = T_2 x_1$, $x_3 = T_1 x_2$, $x_4 = T_2 x_3$ and so on.

Then,

$$d(x_1, x_2) = d(T_1 x_0, T_2 x_1) \leqslant \alpha\, d(x_0, T_1 x_0) + \beta\, d(x_1, T_2 x_1) =$$

$$= \alpha\, d(x_0, x_1) + \beta\, d(x_1, x_2)\,.$$

Therefore,

$$d(x_1, x_2) \leqslant \frac{\alpha}{1-\beta}\, d(x_0, x_1)\,.$$

$$d(x_2, x_3) = d(T_2 x_1, T_1 x_2) \leqslant \alpha\, d(x_2, T_1 x_2) + \beta\, d(x_1, T_2 x_1) =$$

$$= \alpha\, d(x_2, x_3) + \beta\, d(x_1, x_2)\,.$$

Thus,

$$d(x_2, x_3) \leqslant \frac{\beta}{1-\alpha}\, d(x_1, x_2) \leqslant \frac{\beta}{1-\alpha}\, \frac{\alpha}{1-\beta}\, d(x_0, x_1)\,.$$

Similarly,

$$d(x_3, x_4) \leqslant \frac{\alpha}{1-\beta} \frac{\beta}{1-\alpha} \frac{\alpha}{1-\beta} d(x_0, x_1) .$$

In general,

$$d(x_n, x_{n+1}) \leqslant \left(\frac{\alpha}{1-\beta}\right)^{n/2} \left(\frac{\beta}{1-\alpha}\right)^{n/2} d(x_0, x_1) ,$$

when n is even positive integer, and,

$$d(x_n, x_{n+1}) \leqslant \left(\frac{\alpha}{1-\beta}\right)^{n+1/2} \left(\frac{\beta}{1-\alpha}\right)^{n-1/2} d(x_0, x_1) ,$$

when n is an odd positive integer.

For simplicity, put $\alpha/(1-\beta) = k$ and $\alpha\beta/\big((1-\beta)(1-\alpha)\big) = \gamma$ and rewrite the above two inequalities as:

(i) $$d(x_n, x_{n+1}) \leqslant \gamma^{n/2} d(x_0, x_1) ,$$

when n is an even positive integer, and

(ii) $$d(x_n, x_{n+1}) \leqslant k\gamma^{n-1/2} d(x_0, x_1) ,$$

when n is an odd positive integer.

Now, for $m > n$; n both even, we have

$$d(x_n, x_m) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + \dots$$

$$\dots + d(x_{m-2}, x_{m-1}) + d(x_{m-1}, x_m) .$$

$$\leqslant \gamma^{n/2} d(x_0, x_1) + k\gamma^{n/2} d(x_0, x_1) + \gamma^{n/2+1} d(x_0, x_1)$$

$$+ k\gamma^{(n/2)+1} d(x_0, x_1) + \dots + \gamma^{m/2-1} d(x_0, x_1) + k\gamma^{(m/2)-1} d(x_0, x_1) .$$

$$= \gamma^{n/2} d(x_0, x_1) \{1 + \gamma + \gamma^2 + \dots + \gamma^{m-n/2-1}\} .$$

$$+ k\gamma^{n/2} d(x_0, x_1) \{1 + \gamma + \gamma^2 + \dots + \gamma_m{}^{-n/2-1}\} .$$

Thus for $m > n$ and n-even, we have:

$$\text{(iii)} \qquad d(x_n, x_m) \leqslant \frac{\gamma^{n/2}}{1-\gamma} d(x_0, x_1) + \frac{k \cdot \gamma^{n/2}}{1-\gamma} d(x_0, x_1).$$

Similarly for $m > n$ and n-odd, we can have:

$$\text{(iv)} \qquad d(x_n, x_m) \leqslant \frac{k}{1-\gamma} \gamma^{n-1/2} d(x_0, x_1) + \frac{1}{1-\gamma} \gamma^{n+1/2} d(x_0, x_1).$$

Since $\alpha, \beta \geqslant 0$ and $\alpha + \beta < 1$, if follows that

$$\frac{\alpha}{1-\beta} < 1, \quad \frac{\beta}{1-\alpha} < 1 \quad \text{and consequently} \quad \gamma < 1.$$

Therefore, for large n the terms on right hand sides of both the inequalities (iii) and (iv) become arbitrarily small. Thus $\{x_n\}_{n=1}^{\infty}$ is a CAUCHY sequence. Since the space X is complete, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to some point $u \in X$.

Now,

$$d(u, T_1 u) \leqslant d(u, x_n) + d(x_n, T_1 u) = d(u, x_n) + d(T_2 x_{n-1}, T_1 u),$$

where n is chosen to be even positive integer.

Therefore,

$$d(u, T_1 u) \leqslant d(u, x_n) + \alpha\, d(u, T_2 u) + \beta\, d(x_{n-1}, T_2 x_{n-1})$$

or,

$$(1-\alpha)\, d(u, T_1 u) \leqslant d(u, x^n) + \beta\, d(x_{n-1}, x_n)$$

or,

$$d(u, T_1 u) \leqslant \frac{1}{1-\alpha} d(u, x_n) + \frac{\beta}{1-\alpha} d(x_{n-1}, x_n)$$

$$\leqslant \frac{1}{1-\alpha} d(u, x_n) + \frac{\gamma}{k} k \gamma^{n-2/2} d(x_0, x_1),$$

(By inequality (ii))

$$\leqslant \frac{1}{1-\alpha} d(u, x_n) + \gamma^{n/2} d(x_0, x_1).$$

Therefore $d(u, T_1 u) \to 0$, as $n \to \infty$, which gives $T_1 u = u$ i.e., u is a fixed point of T_1.

In the similar way, taking the triangle inequality $d(u, T_2 u) \leqslant d(u, x_n) + d(x_n, T_2 u)$, and n an odd positive integer, we can show that u is a fixed point of T_2.

Thus u is a common fixed point of T_1 and T_2. To show that u is a unique fixed point of T_1, let v be a pcint in X such that $T_1 v = v$. Then,

$$d(u, v) = d(T_2 u, T_1 v) \leqslant \alpha\, d(v, T_1 v) + \beta\, d(u, T_2 u) = 0$$

$u = v$. Thus u is a unique fixed point of T_1. Similarly, we can show that u is a unique fixed point of T_2. Hence u is a common unique fixed point of T_1 and T_2. Hence the theorem. To illustrate the Remark 2.1 and the preceding Theorem, we give a simple example as follows:

Example 2.1. Let T_1, T_2: $[0, 1] \to [0, 1]$, be defined respectively as,

$$T_1 x = X/3\,, \quad x \in [0, 1] \qquad \text{and} \qquad T_2 x = x/4\,, \quad x \in [0, 1]$$

The distance function d is defined in the usual way as $d(x, y) = |x - y|$. The space $X = [0, 1]$ is obviously complete. Now it is easily seen that for the points $x = 1$ and $y = 0$ condition (A) is not satisfied by these mappings for any $\alpha < 1/2$. But on taking $\alpha = 5/8$ and $\beta = 11/30$ so that $\alpha + \beta < 1$, we se that condition (B) is satisfied for all the points in $[0, 1]$ and the common unique fixed point in seen to be zero.

Remark 2.2. By taking $\alpha = \beta$ we get Theorem 1.1 as corollary to Theorem 2.1.

Remark 2.3. If the condition (B) in the previous theorem is not satisfied by T_1 and T_2, but it is satisfied by some iterates T_1^p and T_2^p (p is a positive integer) of T_1 and T_2 respectively, even then the theorem is true.

Thus we have the following general version of the previous theorem.

Theorem 2.2. *Let T_1 and T_2 be two mappings of a complete metric space (X, d) into itself. If there exist non-negative reals α and β, $\alpha + \beta < 1$ and a positive integer p such that,*

(C) $\quad .. d(T_1^p x, T_2^p y) \leqslant \alpha\, d(x, T_1^p x) + \beta\, d(y, T_2^p y) \qquad$ *for all* $\quad x, y \in X$,

where T_1^p and T_2^p stand for p^{th} iterates of T_1 and T_2 respectively, then T_1 and T_2 have a common unique fixed point.

Proof. By the previous theorem, we conclude that T_1^p and T_2^p have a common unique fixed point. Let u be such a point. Now, it follows from a result of CHU and DIAZ [1] that u is a unique fixed point of T_1 as well as of T_2. Hence the theorem.

This Theorem is illustrated by the following example:

Example 2.2. Let $T_1, T_2\colon [0,1] \to [0,1]$ be defined respectively as

$$T_1 x = X/3\,, \quad x \in [0,1] \qquad \text{and} \qquad T_2 x = X/2\,, \quad x \in [0,1]\,.$$

The metric d is defined as $d(x,y) = |x - y|$. We see that condition (B) is not satisfied by T_1 and T_2 for the ponts $x = 0$ and $y = 1$. But it is satisfied by T_1^2 and T_2^2 for all the points in $[0,1]$, when $\alpha = 1/4$ and $\beta = 2/5$. The common unique fixed point of T_1 and T_2 is zero.

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