Riv. Mat. Univ. Parma (3) 1 (1972), 153-163

IOAN POP (*)

Torsional Oscillations of a Porous Infinite Disk under a Circular Magnetic Field. (**)

1. - Introduction.

Recently PAO [1] has studied the boundary layer flow of a viscous, incompressible and electrically conducting fluid over a rotating infinite disk in the presence of a uniform circular magnetic field. He has carried out his analysis for the steady flow. One physical interest in this flow lies in the possibility of using such a field to shield a rotating body from excessive heating.

We have here considered the following problem. A homogeneous, viscous and electrically conducting fluid are in motion over an oscillating porous infinite disk. There is no magnetic field in the distant fluid, but, in the boundary layer, there is an oscillating field in the tangential direction, generated by external means within the disk itself. We have solved the problem by expanding the velocity components in powers of the amplitude oscillation (ε) of the disk. Analytical expressions for velocity, induced magnetic field and current density are obtained as a function of three parameters. The first approximations give only a transverse velocity and a magnetic field which vanishes outside the boundary layer. The second approximations give a radial and an axial velocity component each consisting of a steady and an unsteady part. The steady part of the radial velocity through vanishes at large distance from the disk shows a secondary boundary layer, the thickness of which becomes infinite as the suction parameter k tends to zero. Graphs showing variation of velocity and steady outer flow are plotted.

(*) Indirizzo: Faculty of Mathematics and Mechanics, University of Cluj, Cluj, Romania.

(**) Ricevuto: 5-X-1971.

Similar types of problems with or without the hydromagnetic interaction have been considered in the papers ([2] to [8]).

2. - Equations of motions.

The basic equations governing the unsteady motion of incompressible, viscous and electrically conducting fluids in the presence of a magnetic field are

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla P + (\vec{H}\cdot\nabla)\vec{H} + \nu\Delta\vec{v}\,, \tag{1}$$

$$\nabla\cdot\vec{v} = 0\,, \tag{2}$$

$$\frac{\partial \vec{H}}{\partial t} - \nabla\times(\vec{v}\times\vec{H}) = \nu_m \Delta\vec{H}\,, \tag{3}$$

$$\vec{j} = \nabla\times\vec{H}\,, \qquad \nabla\cdot\vec{B} = 0\,, \qquad \nabla\times\vec{E} = 0\,, \tag{4}$$

where $\vec{v}$ is the fluid velocity vector, $\vec{j}$ the normalized current density vector, $\vec{B}$ the magnetic induction vector, $\vec{E}$ the electric field vector, ν the kinematic viscosity, $\nu_m = 1/(\mu_m\sigma)$ the magnetic diffusivity, μ_m the magnetic permeability and σ the electrical conductivity. We have written $\vec{H} = \vec{H}_1(\mu/\varrho)^{\frac{1}{2}}$ and $P = H^2/2 + P_1/\varrho$ where $\vec{H}_1$ is the magnetic field vector, P_1 the fluid pressure and ϱ the density. In the derivation of (1) and (3) it is assumed that the net charge density is zero and that ν, σ and μ_m are constant.

Let us consider the flow over oscillating porous disk in a fluid otherwise at rest. In addition, an axial electric current of density J_0 is imposed at the disck. Equivalently, the oscillation tangential magnetic field component is imposed at the disk. The cylindrical polar co-ordinates (r, θ, z) with the rotational symmetry of the problem (i.e., $\partial/\partial\theta = 0$) are used. With (u, v, w) and (o, h, o) denoting the components of velocity and of magnetic field strength in the direction of the coordinate lines, the governing equations (1) to (4) become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial P}{\partial r} - \frac{g^2}{r} + \nu\left(\nabla^2 u - \frac{u}{r^2}\right), \tag{5}$$

$$\frac{\partial v}{\partial t} + u\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) + w\frac{\partial v}{\partial z} = \nu\left(\nabla^2 v - \frac{v}{r^2}\right), \tag{6}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} + \nu\nabla^2 w\,, \tag{7}$$

$$\frac{\partial h}{\partial t}+\frac{\partial(uh)}{\partial r}+\frac{\partial(wh)}{\partial z}=\nu_m\left(\nabla^2 h-\frac{h}{r^2}\right), \tag{8}$$

$$\frac{\partial(ru)}{\partial r}+\frac{\partial(rw)}{\partial z}=0, \tag{9}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

The boundary conditions can be written as

$$\begin{cases} u=0, \quad v=\mathrm{Re}\,(r\omega\exp(i\lambda t)) \\ w=-w_0, \quad h=\mathrm{Re}\,(r\Omega\exp(i\lambda t)) \end{cases} \quad \text{at } z=0, \\ u\to 0, \quad v\to 0, \quad h\to 0 \qquad \text{as } z\to\infty, \tag{10}$$

where w_0 is the velocity of suction, λ the frequency of oscillation, ω the characteristic angular velocity and $\Omega=(\mu_m/\varrho)^{1/2}(J_0/2)=\text{const.}$ the corresponding characteristic angular magnetic field strength.

We assume the velocity components, the circular magnetic field and the pressure to be of the following form

$$\begin{cases} u=r\omega F'(\eta,\tau), \quad v=r\omega G(\eta,\tau), \\ w=-w_0-2\omega(2\nu/\lambda)^{\frac{1}{2}}F(\eta,\tau), \quad h=r\Omega H(\eta,\tau), \\ P=2\nu\omega P(\eta,\tau), \end{cases} \tag{11}$$

where $z=(2\nu/\lambda)^{1/2}\eta$, $\tau=\lambda t$ and prime denotes differentiation with respect to η.

By substituting the expressions (11) in the equations of motion (5) to (8) we have

$$\frac{\partial F'}{\partial\tau}+\varepsilon(F'^2-G^2-2FF''+\beta^2H^2)-\frac{\omega_0}{\sqrt{2\nu\lambda}}F''=\frac{1}{2}F''', \tag{12}$$

$$\frac{\partial G}{\partial\tau}+2\varepsilon(F'G-FG')-\frac{\omega_0}{\sqrt{2\nu\lambda}}G=\frac{1}{2}G'', \tag{13}$$

$$\alpha\left(\frac{\partial H}{\partial\tau}-2\varepsilon FH'-\frac{\omega_0}{\sqrt{2\nu\lambda}}H\right)=\frac{1}{2}H'', \tag{14}$$

$$2\frac{\partial F}{\partial\tau}-4\varepsilon FF'=P'+F'', \tag{15}$$

where $\varepsilon = \omega/\lambda$ is the amplitude of oscillation of the plate, $\beta^2 = \Omega/\omega$ the ratio of the ALFVÉN velocity to the fluid velocity at the disk and $\alpha = \nu/\nu_m$ the magnetic PRANDTL number.

The boundary conditions (10) reduce to

$$
\left\{
\begin{array}{llll}
F = F = 0\,, & G = \mathrm{Re}\,(\exp\,(i\tau))\,, & H = \mathrm{Re}\,(\exp\,(i\tau)) & \text{at } \eta = 0\,, \\
F' \to 0\,, & G \to 0\,, & H \to 0 & \text{as } \eta \to \infty\,.
\end{array}
\right. \tag{16}
$$

The equations (12) to (14) together with the boundary conditions (16) are sufficient to determine the functions F, G and H and then the equation (15) on first quadrature will give P.

3. - Solution of the governing equations.

The problem now reduces to the solution of the differential equations (12) to (14) subject to the boundary conditions (16). On the assumption that the amplitude of oscillazion ε to be small, we attempt the solution of these equations by expanding F, G and H as

$$
\left\{
\begin{aligned}
F(\eta, \tau) &= \sum_{n=0}^{\infty} \varepsilon^n F_n(\eta, \tau)\,, \\
G(\eta, \tau) &= \sum_{n=0}^{\infty} \varepsilon^n G_n(\eta, \tau)\,, \\
H(\eta, \tau) &= \sum_{n=0}^{\infty} \varepsilon^n H_n(\eta, \tau)\,.
\end{aligned}
\right. \tag{17}
$$

We shall assume the suction velocity at the disk to be large in comparison with the inflow due to centrifugal effects. This assumption requires

$$
\frac{\omega_0}{\sqrt{2\nu\lambda}} \gg \varepsilon\,,
$$

which for a given amplitude of oscillation is satisfied for either large values of suction or for low values of frequency.

Substitution of the expansion (17) into the fundamental equations (12) to (14), and neglect of terms quadratic in ε, leads to the following three parameter linear partial differential equations set

(18) $$\frac{\partial F_0'}{\partial \tau} - kF_0'' = \frac{1}{2} F_0''' ,$$

(19) $$\frac{\partial G_0}{\partial \tau} - kG_0 = \frac{1}{2} G_0'' ,$$

(20) $$\alpha \left(\frac{\partial H_0}{\partial \tau} - kH_0 \right) = \frac{1}{2} H_0'' ;$$

(21) $$\frac{\partial F_1'}{\partial \tau} + F'^2 - G_0^2 - 2F_0 F_0'' + \beta^2 H_0^2 - kF_0'' = \frac{1}{2} F_1''' ,$$

(22) $$\frac{\partial G_1}{\partial \tau} + 2(F_0' G_0 - F_0 G_0') - kG_1 = \frac{1}{2} G_1'' ,$$

(23) $$\alpha \left(\frac{\partial H_1}{\partial \tau} - 2f_0 H' - kH_0 \right) = \frac{1}{2} H_1'' ,$$

where $k = w_0/(2\nu\lambda)^{\frac{1}{2}}$ is a positive constant which determines the effect of suction.

The boundary conditions (16) reduce to

(24) $$\begin{cases} \left.\begin{array}{l} F_n = F_n' = 0 , \quad G_0 = \mathrm{Re}\,(\exp\,(i\tau)) , \quad H_0 = \mathrm{Re}\,(\exp\,(i\tau)) \\ G_{n+1} = H_{n+1} = 0 \end{array}\right\} \text{ as } \eta = 0 , \\ F_n' \to 0 , \quad G_n \to 0 , \quad H_n \to 0 \qquad \text{at } \eta \to \infty . \end{cases}$$

Primary flow. Solving (18) to (20) we have

(25) $$\begin{cases} F_0(\eta, \tau) = 0 , \\ G_0(\eta, \tau) = \exp\,(-(k + a_1)\eta) \cos\,(\tau - b_1\eta) , \\ H_0(\eta, \tau) = \exp\,(-(\alpha k + a_2)\eta) \cos\,(\tau - b_2\eta) , \end{cases}$$

where

$$a_1 = \left(\frac{\sqrt{k^4 + 4} + k}{2} \right)^{1/2} , \qquad b_1 = \left(\frac{\sqrt{k^4 + 4} - k}{2} \right)^{1/2} ,$$

$$a_2 = \left(\frac{\sqrt{\alpha^4 k^4 + 4\alpha^2} + \alpha^2 k^2}{2} \right)^{1/2} , \qquad b_2 = \left(\frac{\sqrt{\alpha^4 k^4 + 4\alpha^2} - \alpha^2 k^2}{2} \right)^{1/2} .$$

Secondary flow. Now substituting (25) into (21) we obtain

$$(26)\quad \begin{cases} \dfrac{\partial F_1'}{\partial \tau} - \dfrac{1}{2}\exp\left(-2(k+a_1)\eta\right)\left(1+\exp(2i\tau)\cdot\exp(-2ib_1\eta)\right) - \\ -\dfrac{1}{2}\beta^2\exp\left(-2(k+a_2)\eta\right)\left(1+\exp(2i\tau)\cdot\exp(-2ib_2\eta)\right) - kF_1'' = \dfrac{1}{2}F_1''' , \end{cases}$$

with the boundary conditions

$$(27)\qquad F_1(0) = F_1'(0) = 0 , \qquad F_1(\infty) = 0 .$$

From equation (26) we can see that the solution of (26) can be expressed as

$$(28)\qquad F_1(\eta, \tau) = f_1(\eta) + \mathrm{Re}\,[f_2(\eta)\exp(2i\tau)] ,$$

where $f_1(\eta)$ and $f_2(\eta)$ are determined by

$$(29)\qquad \frac{1}{2}f_1''' + kf_1'' = -\frac{1}{2}\exp\left(-2(a_1+k)\eta\right) + \frac{1}{2}\beta^2\exp\left(-2(a_2+k)\eta\right) ,$$

$$(30)\qquad \begin{aligned} \frac{1}{2}f_2''' + kf_2'' + 2if_2' &= -\frac{1}{2}\exp\left(-2(k+a_1+ib_1)\eta\right) - \\ &-\frac{1}{2}\beta^2\exp\left(-2(k+a_1+ib_2)\eta\right) , \end{aligned}$$

with

$$(31)\qquad f_1(0) = f_1'(0) = f_2(0) = f_2'(0) = 0 , \qquad f_1(\infty) = f_2(\infty) = 0 .$$

Only the steady component, $f_1(\eta)$, is considered here because a solution of (30) which satisfies all the boundary conditions may be found even for $k = 0$. Thus the solution of (29) can be written as

$$f(\eta) = \frac{1}{4}\left\{-\frac{1}{a_1(a_1+k)}\exp\left(-2(a_1+k)\eta\right) + \frac{\beta^2}{a_2(a_2+k)}\exp\left(-2(a_2+k)\eta\right) + \right.$$
$$\left. + \left(\frac{1}{a_1(a_1+k)} - \frac{\beta^2}{a_2(a_2+k)}\right)\exp(-2k\eta)\right\} ,$$

$$f_1(\eta) = \frac{1}{8}\left\{\frac{1}{a_1(a_1+k)^2}\exp\left(-2(a_1+k)\eta\right) - \frac{\beta^2}{a_2(a_2+k)^2}\exp\left(-2(a_2+k)\eta\right) - \right.$$
$$\left. -\frac{1}{k}\left(\frac{1}{a_1(a_1+k)} - \frac{\beta^2}{a_2(a_2+k)}\right)\exp(-2k\eta) + \frac{1}{k}\left(\frac{1}{(a_1+k)^2} - \frac{\beta^2}{(a_2+k)^2}\right)\right\} .$$

4. - Discussion.

The dimensionless transverse velocity and the dimensionless circular magnetic field are illustrated against η in Fig. 1 for different values of the suction parameter k and magnetic PRANDTL number α. The continuous curves in

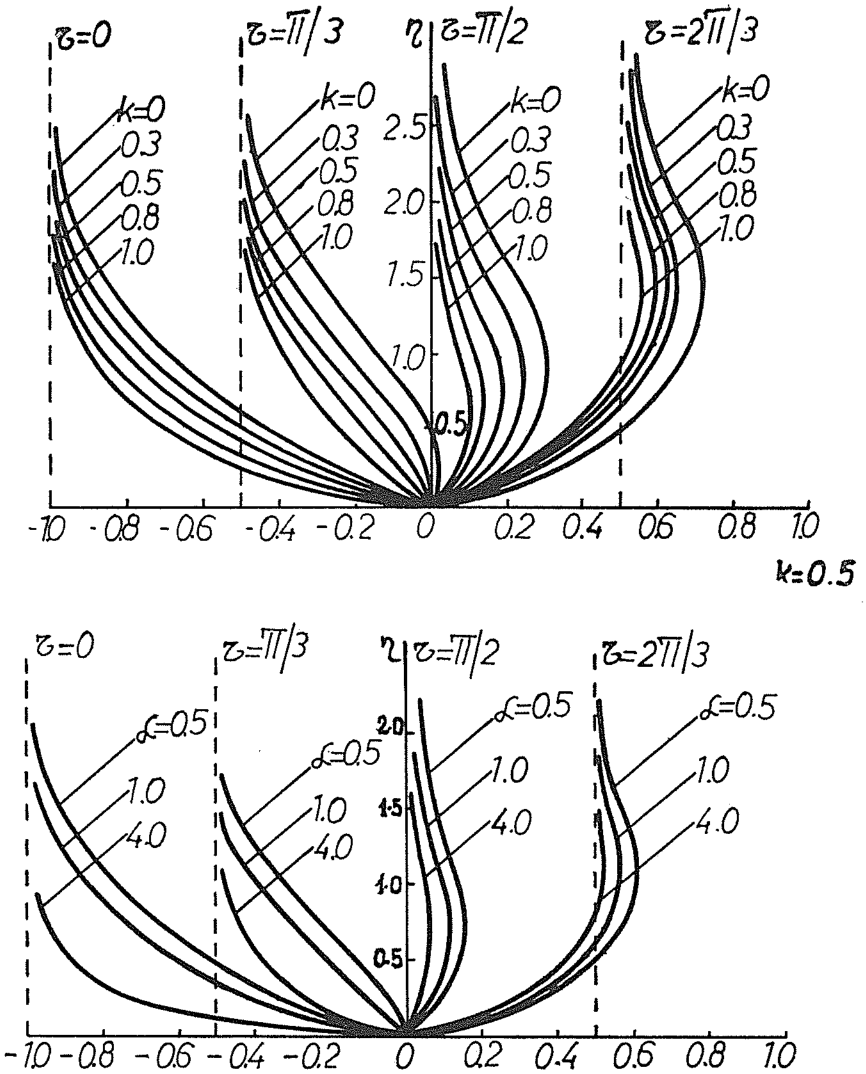

Fig. 1. – Tangential velocity and magnetic field profiles.

these figures represent for selected times, the functions $(G_0 - \cos\tau)$ and $(H_0 - \cos\tau)$, and the vertical straight lines are the values of $\cos\tau$ at the relevant times. It is observed that the velocity and the magnetic field decrease as the both suction and the magnetic PRANDTL number increase. This is what we expect physically, since the effect of the suction is to retard the flow.

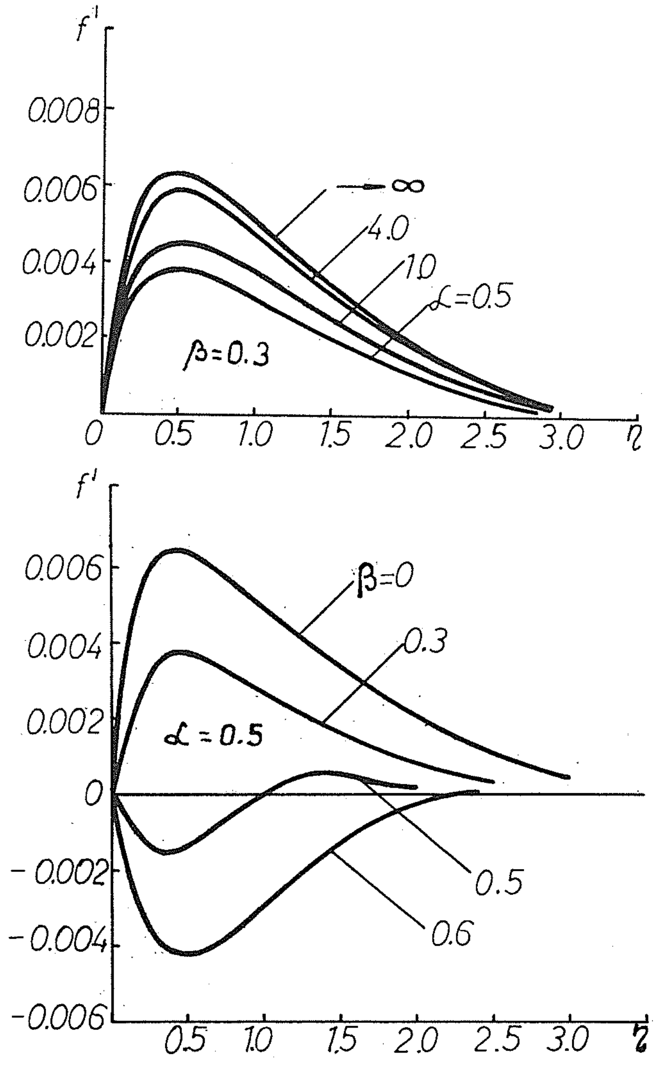

Fig. 2. – Steady radial velocity profiles for $k = 1.0$.

The steady parts of u and w up-to second order of approximation in ε are given by (denote by u_s and w_s respectively)

$$u_s = r\omega\varepsilon f_1'(\eta)\,, \qquad w_s + w_0 = -2\omega\varepsilon(2\nu/\lambda)^{1/2} f_1(\eta)\,.$$

The graphs of steady components of radial and axial flows are plotted against η for the suction parameter $k = 1.0$ and for various values of α and β and are shown in Fig. 2 and 3. It will be found from these figures

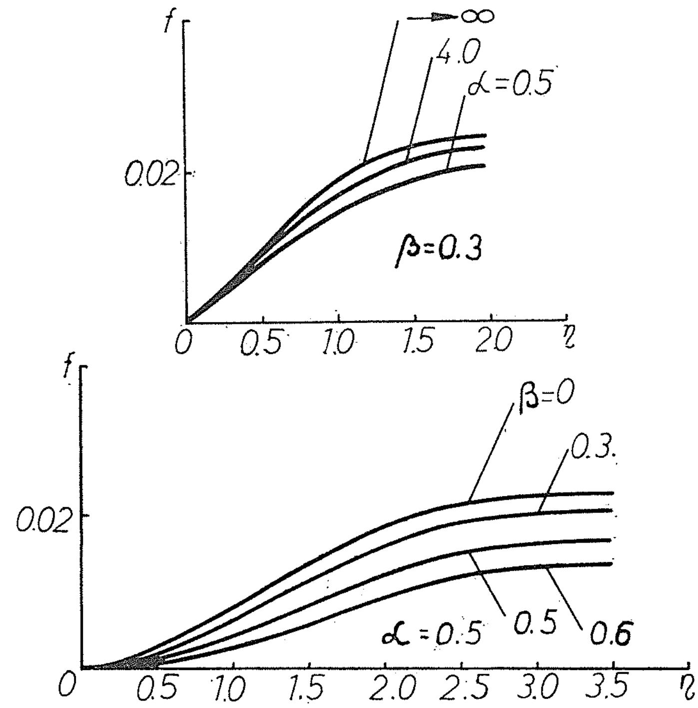

Fig. 3. – Steady axial velocity profiles for $k = 1.0$.

that the magnitude of the radial velocity decreases rapidly and so becomes negligible outside the boundary layer and that of the axial velocity tends to a finite negative value for large values of η, which is required for continuity of the flow. Also the maximum values of the steady radial and axial velocity components are found to decrease with the increase of the magnetic field.

It is noted that in the special case of $\alpha = 1$ and $\beta = 1$ the radial and axial flow are zero. Therefore for each value of the suction parameter k there are critical values of the pair (α, β) above which no steady flow is possible.

The steady inflow as $\eta \to \infty$ due to the oscillations can be expressed as

$$-\frac{w_{s,\infty} + w_0}{2\varepsilon^2 \sqrt{2\nu\lambda}} = \frac{1}{8k}\left\{\frac{1}{(a_1 + k)^2} - \frac{\beta^2}{(a_2 + k)^2}\right\}.$$

This quantity is displayed in Fig. 4 for various values of k, α and β. It is clear that the inflow due to the oscillations decreases as the wall suction increases. The solution is, of course, singular as $k \to 0$. This happens due to the fact that we have neglected the convective part of the inertia force as compared to the centrifugal one, which is not justified at a large distance

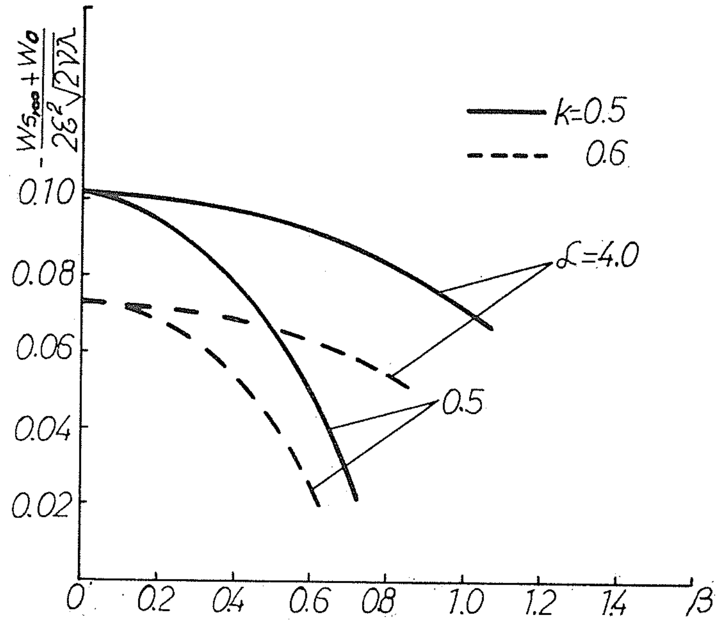

Fig. 4. – Steady axial flow.

from the disk. It is also observed that the steady flow decreases as α and β increase.

Another interesting feature of the flow is the distribution of the electrical current density which is given by

$$j_r = -(\varrho/\mu_m)^{1/2}\frac{\partial g}{\partial z} = -(J_0/2)(\omega/\nu)^{1/2}\, rH'(\eta, \tau)\,,$$

$$j_\theta = 0\,,$$

$$j_z = (\varrho/\mu_m)^{1/2}\frac{1}{r}\frac{\partial (gr)}{\partial r} = J_0 H(\eta, \tau)\,,$$

where (j_r, j_θ, j_z) are the components of the current density in the (r, θ, z) directions. The current density is zero at a great distance from the disk.

The tangential component of the viscous shearing stress at the disk is given by

$$\tau_{z\theta} = \varrho\nu\left(\frac{\partial v}{\partial z}\right)_{z=0} = -\,\varrho r\omega\,\sqrt{\frac{\lambda\nu}{2}}\,A\cos(\tau+\gamma)\,,$$

where

$$A = \sqrt{(a_1+k)^2+b_1^2}\,,\qquad \gamma = \mathrm{tg}^{-1}\frac{b_1}{a_1+k}\,.$$

The transverse shearing stress has a phase lead over the oscillation of the disk and this lead decreases with increasing suction.

References.

[1] H. P. Pao, *Magnetohydrodynamic flows over a rotating disk*, AIAA J. **6** (1968) 1285-1291.

[2] S. Rosenblat, *Torsional oscillations of a plane in a viscous fluid*, J. Fluid Mech. **6** (1959), 206-220.

[3] N. Datta, *Torsional oscillation of an infinite plate in a conducting fluid under a transverse magnetic field*, J. Phys. Soc. Japan **19** (1964), 2219-2223.

[4] S. Datta, *Oscillatory hydromagnetic disk flow*, Appl. Sci. Res. **11** B (1964), 43-48.

[5] R. E. Kelly, *Torsional oscillations of a plate in magnetohydrodynamics*, J. Appl. Math. Mech. (trans. ASME) **31** E (1964), 547-548.

[6] R. E. Kelly, *On the torsional oscillations of a plane with large suction*, Z. Angew. Math. Phys. **16** (1965), 516-519.

[7] P. V. Subba Raju, *Hydromagnetic flow due to the torsional oscillations of a porous infinite plate*, J. Phys. Soc. Japan, **21** (1966), 2383-2390.

[8] I. Pop, *Le mouvement instationnaire à succion d'un fluide visqueux incompressible sous l'action d'un disque tournant*, Comptes Rendus Ac. Sci. Paris **268** A (1969), 228-230.

Summary.

The flow of a viscous, electrically and conducting fluid due to an infinite porous oscillating disk in the presence of a circular magnetic field, is considered. The derived fundamental equations are solved by an expansion method in power series of the amplitude of oscillation of the disk. Numerical results are given for the velocity profiles, magnetic field, skin-friction and steady inflow.

* * *