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On [R, log $m \cdot$ log n, 2] Summability of Double Fourier Series. (**)

1. - Let $f(x, y)$ be L in the square $[-\pi, \pi; -\pi, \pi]$ and be periodic in each variable with period 2π. The partial sum of double Fourier series associated with $f(x, y)$ is given, as usual, by

$$s_{\mu\nu} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi(u,\, v) \frac{\sin\{(\mu + \frac{1}{2})u\}}{2\sin(u/2)} \frac{\sin\{(\nu + \frac{1}{2})v\}}{2\sin(v/2)} \,\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{\varphi(u,\, v)}{uv} \sin(\mu u) \sin(\nu v) \,\mathrm{d}u\,\mathrm{d}v + o(1)\,,$$

where

$$\varphi(u,\, v) = \tfrac{1}{4}[f(x+u,\, y+v) + f(x+u,\, y-v) + f(x-u,\, y+v) + f(x-u,\, y-v) - 4s]\,.$$

Definition. The double series $\sum\sum A_{\mu\nu}$ is said to be summable by strong logarithmic means of positive order k or summable [R, log m, log n, k], prov-

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ided there exists some constant s such that

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |s_{\mu\nu} - s|^k/(\mu\nu) = o(\log m \log n) ,$$

where

$$S_{mn} = \sum_{1}^{m} \sum_{1}^{n} A_{\mu\nu} .$$

Here in this report we shall be concerned with $k = 2$. We shall prove the following theorem.

Theorem A. *If $f(x, y) \in L^2$ and if for some fixed τ and δ we have:*

$$\int_s^\tau \int_t^\delta \{\varphi(u, v)\}^2/(u, v)\, du\, dv = o(\log s^{-1} \log t^{-1}) , \tag{1.1}$$

$$\int_0^\pi du \,\Big| \int_t^\delta \{\varphi(u, v)\}^2/v\, dv\Big| = o(\log t^{-1}) , \tag{1.2}$$

$$\int_0^\pi dv \Big| \int_s^\tau \{\varphi(u, v)\}^2/u\, du\Big| = o(\log s^{-1}) , \tag{1.3}$$

then

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |s_{\mu\nu} - s|^2(\mu\nu) = o(\log m \log n) .$$

It can be easily seen that

$$\Big[\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |s_{\mu\nu} - s|^2/(\mu\nu) = o(\log m \log n)\Big] \Rightarrow$$

$$\Rightarrow \Big[\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |s_{\mu\nu} - s|^k/(\mu\nu) = o(\log m \log n)\Big] , \qquad \forall \text{ positive } k \leqslant 2 .$$

For if $k < 2$ we have

$$\begin{aligned}\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |s_{\mu\nu} - s|^k/(\mu\nu) &= \sum \sum |s_{\mu\nu} - s|^k/(\mu^{1-k/2}\ \nu^{1-k/2})\ (\mu^{k/2}\ \nu^{k/2}) \\ &\leqslant \Big(\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |s_{\mu\nu} - s|^2/\mu\nu\Big)^{k/2}\Big(\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (1/\mu\nu)\Big)^{1-k/2} \\ &\leqslant o(\log^{k/2} m \log^{k/2} n)\ (\log^{1-k/2} m \log^{1-k/2} n) \\ &= o(\log m \log n) .\end{aligned}$$

2. – In 1913, HARDY and LITTLEWOOD proved the following theorem on strong summbility (c, 2) or summability [c, 2].

Theorem B. If $f(t)$ is of integrable square in a neighbourhood of x and if

$$\int_0^t \{\varphi(u)\}^2 \mathrm{d}u = o(t) ,$$

then

$$\sum_{\mu=1}^{n} |s_\mu - s| = o(n) .$$

This theorem was generalised in every direction by various workers including the authores of the Theorem B themselves. HSÜ [**1**] generalised the process of [c, k] summability to the case of double FOURIER series, SHARMA [**2**] further extended the result of HSÜ by replacing the ordinary partial sum by CÈSARO mean of positive order. Here in our present report Theorem A is an extension of a theorem by SINGH [**3**] on [R, log n, 2] summability to the case of a double FOURIER series. SINGH [**3**] proves the following theorem.

Theorem C. If $f \in L^2$ and if, for some fixed δ, we have

$$\int_t^\delta \{\varphi(u)\}^2/u \, \mathrm{d}u = o(\log t^{-1}) ,$$

then

$$\sum_{\nu=1}^{n} \{s_\nu - s\}^2/\nu = o(\log n) .$$

For the proof of Theorem A we shall require the following lemmas.

Lemma 1 (a). *If* (1.1) *is true, then*

$$\int_0^s \int_0^t |\varphi(u, v)| \, \mathrm{d}u \, \mathrm{d}v = o\big(st\sqrt{\log s^{-1} \log t^{-1}}\big)$$

Proof. It can be easily shown that

$$(1.1) \quad \Rightarrow \int_0^s \int_0^t \{\varphi(u, v)\}^2 \mathrm{d}u \, \mathrm{d}v = o(st \log s^{-1} \log t^{-1}) .$$

Now using SCHWARTZ inequality, we get

$$\int_0^s \int_0^t \varphi(u, v)\, du\, dv \leqslant \Big(\int_0^s \int_0^t \{\varphi u, v)\}^2 du\, dv\Big)^{1/2} \Big(\int_0^s \int_0^t du\, dv\Big)^{1/2}$$

$$= o\big(st\sqrt{\log s^{-1} \log t^{-1}}\big)\,.$$

Lemma 1 (b).

$$\Rightarrow \int_0^\pi du \Big|\int_0^t \varphi(u, v)\, dv\Big| = o\big(t\sqrt{\log t^{-1}}\big)\,. \tag{1.2}$$

Proof of the Lemma 1(b) is same as that of Lemma 1(a).

Lemma 2. *If*

$$0 \leqslant s \leqslant \tau < \pi\,, \qquad 0 \leqslant u \leqslant \tau < \pi\,, \qquad s \neq u\,,$$

$$0 \leqslant t \leqslant \delta < \pi\,, \qquad 0 \leqslant v \leqslant \delta < \pi\,, \qquad t \neq v\,,$$

then

$$\Big|\sum_{\nu=1}^{n} \frac{\sin(\nu t)\sin(\nu v)}{\nu}\Big| \leqslant K \log \frac{t+v}{|t-v|}\,,$$

where K depends on δ only, etc. .

Proof.

$$\sum_{\nu=1}^{n} \frac{\sin(\nu t)\sin(\nu v)}{\nu} = \frac{1}{2}\sum_{\nu=1}^{n} \frac{\cos \nu(t-v) - \cos \nu(t+v)}{\nu}$$

$$= \frac{1}{2}\int_{|t-v|}^{t+v} \Big(\sum_{\nu=1}^{n} \sin(\nu\theta)\Big)\, d\theta = \frac{1}{4}\int_{|t-v|}^{t+v} \frac{\cos(\theta/2) - \cos(n+\frac{1}{2})\theta}{\sin(\theta/2)}\, d\theta\,.$$

Since $|\cos(\theta/2) - \cos(n+\frac{1}{2})\theta| \leqslant 2$ and since for $0 \leqslant \theta \leqslant 2\delta < 2\pi$, $\sin(\theta/2) > K\theta$ and that $K > 0$ depends on δ only, the result follows readily.

Proof of Theorem A. Making usual standard simplifications by taking $f(x, y)$ to be an even-even function, and assuming at $x = 0$, $y = 0$, $s = 0$, we can write

$$s_{\mu\nu} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{f(u, v)}{\sin(u/2)\sin(v/2)} \sin(\mu + \tfrac{1}{2})u \sin(\nu + \tfrac{1}{2})v \, du\, dv$$

$$= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi f(u, v)\frac{\sin(\mu u)}{v}\frac{\sin(\nu v)}{v}\, du\, dv + o(1)$$

$$= \frac{1}{\pi^2}\Big(\int_0^\tau\int_0^\delta + \int_\tau^\pi\int_0^\delta + \int_0^\tau\int_\delta^\pi + \int_\tau^\pi\int_\delta^\pi\Big) + o(1)\,,$$

(3.1) $$s_{\mu\nu} = \frac{1}{\pi^2}(I_1 + I_2 + I_3 + I_4) + o(1)\,, \qquad \text{say}$$

where

(3.2) $$I_4 = o(1)\,,$$

by Riemann-Lebesgue theorem

$$I_1 = \Big(\int_0^{1/m}\int_0^{1/n} + \int_{1/m}^{\tau}\int_0^{1/n} + \int_0^{1/m}\int_{1/n}^{\delta} + \int_{1/m}^{\tau}\int_{1/n}^{\delta}\Big)$$

$$= I_{1,1} + I_{2,2} + I_{1,3} + I_{1,4}\,, \qquad \text{say}$$

where

$$I_{1,1} = \int_0^{1/m}\int_0^{1/n} + \frac{f(u, v)}{uv}\sin(\mu u)\sin(\nu v)\, du\, dv\,,$$

$$\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} I_{1,1}^2/(\mu\nu) = \Big(O\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} 1/(\mu\nu)\ \Big[\int_0^{1/m}\int_0^{1/n} f(u, v)\{(\mu u\cdot\nu v)/uv\}\, du\, dv\Big]^2\Big)$$

$$= O\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}\mu\nu\Big)\cdot o\big(1/(m^2 n^2)\log m \log n\big)\,,$$

(3.3) $$\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} I_{1,1}^2/(\mu\nu) = o(\log m \log n) \qquad \text{by (1.1)}$$

and

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_{1,2}^2/(\mu\nu) = O\Big(\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \nu/\mu\Big[\int_{1/m}^{\tau} \int_{0}^{1/n} |f(u, v)|\, \mathrm{d}u\, \mathrm{d}v\Big]^2\Big)$$

$$= o\Big(\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \nu/\mu[1/n^2 \log n]\Big)\,,$$

by (1.2) .

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_{1,2}^2/(\mu\nu) = o(\log m \log n) \tag{3.4}$$

Similarly

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_{1,3}^2/(\mu\nu) = o(\log m \log n)\,. \tag{3.5}$$

Consider the integral $I_{1,4}$, we have

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_{1,4}^2/(\mu\nu) = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} 1/(\mu\nu) \int_{1/m}^{\tau} \int_{1/n}^{\delta} \{f(u, v)/(uv)\} \sin(\mu u) \sin(\nu v)\, \mathrm{d}u\, \mathrm{d}v \cdot$$

$$\cdot \int_{1/m}^{\tau} \int_{1/n}^{\delta} \{f(s, t)/(st)\} \sin(\mu s) \sin(\nu t)\, \mathrm{d}s\, \mathrm{d}t$$

$$= \int_{1/m}^{\tau} \int_{1/n}^{\delta} \{f(u, v)/(uv)\}\, \mathrm{d}u\, \mathrm{d}v \int_{1/m}^{\tau} \int_{1/n}^{\delta} f(s, t)/(st) \Big\{\sum_{\mu=1}^{m} \sin(\mu u) \sin(\mu v)\, \mu^{-1}\Big\} \cdot$$

$$\cdot \Big\{\sum_{\nu=1}^{n} \sin(\nu v) \sin(\nu t)\, \nu^{-1}\Big\}\, \mathrm{d}s\, \mathrm{d}t\,,$$

$$\left\{ \begin{aligned} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_{1,4}^2/(\mu\nu) \leqslant K \int_{1/m}^{\tau} \int_{1/n}^{\delta} \{f(u, v)/(uv)\, \mathrm{d}u\}\, \mathrm{d}v \int_{1/m}^{\tau} \int_{1/n}^{\delta} \{f(s, t)/(st)\} \cdot \\ \cdot \log \frac{s+u}{|s-u|} \log \frac{t+v}{|t-v|}\, \mathrm{d}s\, \mathrm{d}t \end{aligned} \right. \tag{3.6}$$

by Lemma (2).

Since the function integrated out in (3.6) is symmetric in s&u and t&v, the contribution of domain of integration for which $s < u$, $t < v$, should be equal to the contribution of portion for which $s > u$, $t > v$, hence (3.6) is equal to

$$
(3.7)\quad \begin{cases}
4k \int\limits_{1/m}^{\tau} \int\limits_{1/n}^{\delta} \frac{f(u, v)}{uv} \, du \, dv \int\limits_{u}^{\tau} \int\limits_{v}^{\delta} \frac{f(s, t)}{st} \log \frac{s+u}{|s-u|} \log \frac{t+v}{|t-v|} \, ds \, dt \\
= 4k \int\limits_{1/m}^{\tau} \int\limits_{1/n}^{\delta} \frac{f(u, v)}{uv} \, du \, dv \int\limits_{1}^{\tau/u} \int\limits_{1}^{\delta/v} \frac{f(u\xi, v\eta)}{\xi\eta} \log \frac{\xi+1}{|\xi-1|} \log \frac{\eta+1}{|\eta-1|} \, d\xi \, d\eta \\
= 4k \int\limits_{1}^{m\tau} \int\limits_{1}^{n\delta} \frac{1}{\xi\eta} \log \frac{\xi+1}{|\xi-1|} \log \frac{\eta+1}{|\eta-1|} \, d\xi \, d\eta \int\limits_{1/m}^{\tau/\xi} \int\limits_{1/n}^{\delta/n} \frac{f(u, v) f(u\xi, v\eta)}{uv} \, du \, dv \,.
\end{cases}
$$

The interior integral does not exceed

$$
\left(\int\limits_{1/m}^{\tau/\xi} \int\limits_{1/n}^{\delta/\eta} \frac{\{f(u, v)\}^2}{uv} \, du \, dv \right)^{1/2} \quad \left(\int\limits_{1/m}^{\tau/\xi} \int\limits_{1/n}^{\delta/\eta} \frac{\{f(u\xi, v\eta)\}^2}{uv} \right)^{1/2} ,
$$

$$
(3.8)\qquad \int\limits_{1/m}^{\tau} \int\limits_{1/n}^{\delta} \frac{\{f(u, v)\}^2}{uv} \, du \, dv = o(\log m \log n) \,.
$$

Combining (3.7) and (3.8) we see that (3.7) is $= o(\log m \log n)$ hence

$$
(3.9)\qquad \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_{1,4}^2/(\mu\nu) = o(\log m \log n) \,.
$$

In view of (3.3), (3.4), (3.5) and (3.9) and MINKOWSKI's inequality we find that

$$
(3.10)\qquad \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_1^2/(\mu\nu) = o(\log m \log n) \,.
$$

Now we proceed to consider integral I_2 of (3.1)

$$
I_2 = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \frac{1}{\mu\nu} \left[\int\limits_{\tau}^{\pi} \int\limits_{1/n}^{\delta} + \int\limits_{\tau}^{\pi} \int\limits_{0}^{1/} \right] \frac{f(u, v)}{uv} \sin(\mu u) \sin(\nu v) \, du \, dv
$$

$$
(3.11)\qquad I_2 = I_{2,1} + I_{2,2}, \qquad \text{say.}
$$

We have for $I_{2,1}$ (see SHARMA [3])

$$(3.12)\quad \begin{cases} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_{2,1}^2/(\mu\nu) = O\left(\sum_{\mu=1}^{m} \frac{1}{\mu} \left[\int_{1/n}^{\delta} \frac{f(u,v)}{v} \, dv \int_{1/n}^{\delta} \frac{f(u,t)}{t} \log \frac{t+v}{|t-v|} \, dt \right] \right. \\ \qquad = O\left(\sum_{\mu=1}^{m} \frac{1}{\mu} \cdot L \right), \end{cases}$$

where

$$L = \int_{1/n}^{\delta} \frac{f(u,v)}{v} \, dv \int_{1/n}^{\delta} \frac{f(u,t)}{t} \log \frac{t+v}{|t-v|} \, dt \, .$$

Arguing as in case of $I_{1,4}$ we easily see that

$$L = o\left[\log n \int_{1}^{n\delta} \frac{1}{\eta} \log \frac{1+\eta}{|\eta - 1|} \, d\eta \right],$$

$$(3.13)\qquad L = o(\log n) \, .$$

Combining (3.12) and (3.13) we have

$$(3.14)\qquad \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_{2,1}^2/(\mu\nu) = o(\log m \log n) \, .$$

Also

$$I_{2,2} = \int_{\tau}^{\pi} \int_{0}^{1/n} \frac{f(u,v)}{uv} \sin(\mu u) \sin(\nu v) \, du \, dv$$

or

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_{2,2}^2/(\mu\nu) = O\left(\sum_{1}^{m} \sum_{1}^{n} \frac{\nu}{\mu} \, o\left(\frac{1}{n^2} \log n \right) \right),$$

$$(3.15)\qquad = o(\log m \log n) \, .$$

From (3.14) and (3.15) we find that

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_2^2/(\mu\nu) = o(\log m \log n) . \tag{3.16}$$

Proceeding as in case of I_2 we can write

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I_3^2/(\mu\nu) = o(\log m \log n) . \tag{3.17}$$

Applying MINKOWSKI's inequality for (3.1) and making use of (3.2), (3.10), (3.16) and (3.17), we prove that

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} I^2/(\mu\nu) = o(\log m \log n) .$$

This completes the proof of the Theorem A.

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