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The Vitali-Hahn-Saks Theorem for Operator-Valued Linear Mappings. (**)

The VITALI-HAHN-SAKS theorem is a result from classical measure theory concerning the convergence properties of a sequence of measures.

Let (X, Σ, μ) be a finite measure space (real or complex). Suppose that $\{\nu_n\}$ is a sequence of μ-continuous finite measures such that, for every E in Σ, $\nu_n(E)$ converges to a finite number $\nu(E)$. Then:

(a) The measure ν_n are *uniformly μ-absolutely continuous*; i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\mu(E)| < \delta$ implies $|\nu_n(E)| < \varepsilon$ for all n;

(b) If $\{E_m\} \subseteq \Sigma$ is a sequence of sets such that $E_m \downarrow E_0$ where E_0, is a set of μ-measure zero, then for any $\varepsilon > 0$, there is p_0 such that $|\nu_n(E_p)| < \varepsilon$ for $p \geqslant p_0$ and for all n;

(c) The set function ν is a measure on Σ.

For references on this theorem, see DUNFORD and SCHWARTZ ([5]), p. 158, theorem 2; p. 159, corollary 3; p. 160, corollary 4) and ZAANEN ([7], p. 332, theorem 4).

This theorem has been extended to non-commutative integration theory by J. AARNES [1] and C. AKEMANN [2]. In their results the measures are replaced by normal linear functionals on a VON NEUMANN algebra M. Positive normal functionals are similar to positive measures in that they have the prop-

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erty of complete additivity, similar to the countable additivity property of measures.

Let $\{f_n\}$ be a sequence of normal linear functionals on a VON NEUMANN algebra M. Suppose that $f_n(P)$ converges to a finite number $f(P)$ for every projection P in M. Then AARNES and AKEMANN proved the following:

(a) $\lim f_n(A) = f(A)$ exists for every A in M;

(b) The norms of the f_n's are uniformly bounded;

(c) f is a normal linear functional on M.

In this work, we replace the normal linear functionals by normal linear mappings Φ_n from M into a second VON NEUMANN algebra N. Assuming that $\lim \Phi_n(P)$ exists in norm for every projection P in M, we show that $\lim \Phi_n(A) = \Phi(A)$ exists in norm for every A in M and that Φ is a normal linear mapping from M to N (Theorem 4).

We use a result of AARNES (Theorem 1) to obtain a type of uniform boundedness theorem (Theorem 3) useful in the proofs of Theorem 4 and 5. In Theorem 5, we apply Theorem 4 to the special case when the Φ_n's are expectation mappings.

I. - Notation, definitions and preliminaries.

Throughout we assume that M and N are two VON NEUMANN algebras contained in $\mathfrak{L}(\mathcal{H})$, the set of all bounded linear operators on a HILBERT space $\mathcal{H}$. If L is any von NEUMANN algebra, by L_* we will mean that unique BANACH space whose dual space can be identified (as a BANACH space) with L; see S. SAKAI [**6**].

A linear map Φ from M to N is called *completely additive* if whenever $\{P_\alpha\}$ is a disjoint collection of projections in M with sum ΣP_α, then $\Phi(\Sigma P_\alpha) = \Sigma \Phi(P_\alpha)$.

A linear map Φ from M to N is called *normal* if for every increasing directed set $\{A_\alpha\}$ of positive operators in M with $\underset{\alpha}{\text{l.u.b.}}\ A_\alpha = A$, then $\Phi(A) = \underset{\alpha}{\text{l.u.b.}}\ \Phi(A_\alpha)$.

We note that if Φ is normal, then it is completely additive.

Suppose that $N \subseteq M$ and that N contains the identity map I. A linear map Φ is an *expectation mapping of M on N* if

(a) $\Phi(I) = I$;
(b) $A\Phi(B) = \Phi(AB)$ for all A in N and B in M;
(c) Φ is adjoint preserving;
(d) Φ is positive.

This is a generalization of the classical probabilistic notion of expectation mappings; for more on these abstract expectation mappings, see, for example, A. DE KORVIN [**3**].

Let $\mathcal{F}$ be a collection of continuous linear maps from M to N. The collection $\mathcal{F}$ is called *uniformly weakly completely additive* if the following condition holds:

Suppose that $\{P_\alpha\}$ is a downward directed commutative collection of projections in M with $\underset{\alpha}{\text{g.l.b.}}\ \{P_\alpha\} = 0$ and suppose that x is in $\mathcal{H}$. Then for every $\varepsilon > 0$, there is a α_0 such that $|(f(P_\alpha)x, x)| < \varepsilon$ for all f in $\mathcal{F}$ and $\alpha \geqslant \alpha_0$.

We now state the two main theorems of AARNES for reference.

Theorem 1. *If $\mathcal{F}$ is a family of normal linear functionals on M and if $\mathcal{F}$ is pointwise bounded on the projections in M, then $\mathcal{F}$ is uniformly bounded on norm-bounded subsets of M.*

Theorem 2. *Let $\{f_n\}$ be a sequence of normal linear functionals on M such that* $\lim f_n(P)$ *exists for every projection P in M. Let $f(P)$ denote the limit of $f_n(P)$ for every projection P. Then:*

(a) *f can be extended to all of M as a continuous linear functional;*

(b) *f is a normal linear functional.*

Moreover, if L is a commutative von Neumann subalgebra of M and if $\mathcal{P}$ denotes the set of projections in M, then

(c) *the maps $\{g_n = f_n | L \cap P\}$ form an equicontinuous family at 0 for the topology $\sigma(M, M_*) | L \cap P$.*

II. - The main results.

The following theorem is a type of uniform boundedness theorem similar to Theorem 1. We recall that since $(N_*)^* = N$, we can therefore consider N_* as a subset of N^*.

Theorem 3. *Let $\mathcal{F}$ be a collection of continuous linear mappings from M into N with the property that $\varrho \circ \Phi$ is in M_* for every ϱ in N_* and Φ in $\mathcal{F}$. Suppose $\mathcal{F}$ is pointwise bounded on the projections of M. Then $\mathcal{F}$ is uniformly bounded on norm-bounded subsets of M.*

Proof. Let P denote any projection in M. Then by hypothesis there is a number $M_P > 0$ such that $\|\Phi(P)\| \leqslant M_P$ for every Φ in $\mathcal{F}$. Now let ϱ

be in N_*. Denoting $\varrho \circ \Phi$ by ϱ_Φ, we have

$$|\varrho_\Phi(P)| \leqslant \|\varrho\| \, \|\Phi(P)\| \leqslant \|\varrho\| M_P = M_{P,\varrho} .$$

For a fixed ϱ in N_* each ϱ_Φ is in M_* by hypothesis; so by Theorem 1 the collection $\mathcal{F}' = \{\varrho_\Phi \mid \Phi \in \mathcal{F}\}$, being pointwise bounded on the projections in M, is uniformly bounded on norm-bounded subsets of M. If B denotes such a set, then

$$|\varrho(\Phi(x))| \leqslant M_\varrho$$

for every Φ in $\mathcal{F}$ and x in B.

Thinking of $\Phi(x)$ as a continuous linear function $\widetilde{\Phi(x)}$ on N_*, we then have

$$|\widetilde{\Phi(x)}(\varrho)| \leqslant M_\varrho$$

for every Φ in $\mathcal{F}$ and x in B. This inequality holds true for every ϱ in N_*. Hence by the Uniform Boundedness Principle we find that the norms of the mappings $\widetilde{\Phi(x)}$ for Φ in $\mathcal{F}$ and x in B are uniformly bounded; i.e., there is an $M_0 > 0$ such that $\|\widetilde{\Phi(x)}\| \leqslant M_0$ for all Φ in $\mathcal{F}$ and x in B. But $\|\Phi(x)\| = \|\widetilde{\Phi(x)}\|$ since $\Phi(x)$ considered either as an element of N or as a mapping on N_* has the same norm. Thus $\mathcal{F}$ is uniformly bounded on B. ∎

Note that the hypothesis of Theorem 3 is fulfilled if each Φ in $\mathcal{F}$ is a continuous normal linear mapping.

We now state the main theorem, which is a generalization of Theorem 2 and our generalization of the VITALI-HAHN-SAKS theorem.

Theorem 4. *Let $\{\Phi_n\}$ be a sequence of continuous linear mappings from M into N. Suppose that $\varrho \circ \Phi_n$ is in M_* for every Φ_n and for every ϱ in N_*. Suppose that* $\lim \Phi_n(P) = \Phi(P)$ *exists (in the norm topology of N) for every projection P in M. Then:*

(a) *Φ can be extended to a continuous linear map from M into N. If $\mathfrak{T}$ denotes the set of projections in M and if L is a commutative von Neumann subalgebra of M, let $\psi_n = \Phi_n | \mathfrak{T} \cap L$. Then:*

(b) *the collection $\{\psi_n\}$ is equicontinous at 0 with respect to the weak operator topologies on $\mathfrak{T} \cap L$ and N;*

(c) *the collection $\{\Phi_n\}$ is uniformly weakly completely additive.*
If each Φ_n is positive, then

(d) *Φ_n is positive, normal, and completely additive.*

Proof. For part (a), it is clear that Φ can be extended to finite linear combinations of projections. To show that Φ can be extended to all of M it is sufficient to show that Φ can be extended to the set of self-adjoint elements of M. Since each element in M is of the form $A_1 + iA_2$, where A_1 and A_2 are self-adjoint, by linearity Φ can then be extended to M.

From the general theory of VON NEUMANN algebras, every self-adjoint A in M is the limit in norm of a sequence $\{F_n\}$, where each F_n is a finite linear combination of projections in M; see, e.g., ([4], corollary 2, p. 4).

We now show that $\{\Phi_n(A)\}$ is a CAUCHY sequence in the BANACH space N and we will define the resulting limit as $\Phi(A)$.

First we note that by the convergence of $\Phi_n(P)$ for every projection P, we have by Theorem 3 that $\{\Phi_n\}$ is uniformly bounded on norm-bounded sets; hence there is an $M_0 > 0$ such that $\|\Phi_n\| \leqslant M_0$ for all n. Thus

$$\begin{aligned}\|\Phi_n(A) - \Phi_m(A)\| &= \|(\Phi_n - \Phi_m)(F_p) + (\Phi_n - \Phi_m)(A - F_p)\| \\ &\leqslant \|(\Phi_n - \Phi_m)(F_p)\| + \|(\Phi_n - \Phi_m)(A - F_p)\| \\ &\leqslant \|(\Phi_n - \Phi_m)(F_p)\| + 2M_0\|A - F_p\| .\end{aligned}$$

For $\varepsilon > 0$, choose p_0 such that $\|A - F_p\| < \varepsilon/4M_0$ for all $p \geqslant p_0$. For this p_0 choose m_0 and n_0 such that $\|(\Phi_n - \Phi_m)(F_{p_0})\| < \varepsilon/2$ for all $m \geqslant m_0$ and $n \geqslant n_0$. We then have for the given ε

$$\|\Phi_n(A) - \Phi_m(A)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $m \geqslant m_0$ and $n \geqslant n_0$. Thus $\{\Phi_n(A)\}$ is a CAUCHY sequence and the extended map Φ is easily seen to be linear. The continuity of Φ follows directly from the fact that the sequence $\{\Phi_n\}$ is uniformly bounded on norm-bounded sets. Thus (a) is proved.

To prove (b), we first prove that the collection $\{\psi_n\}$ is equicontinuous with respect to the topologies $\sigma(M, M_*) \,|\, \mathfrak{I} \cap L$ and $\sigma(N, N_*)$.

A typical basic $\sigma(N, N_*)$-neighborhood of 0 is of the form $\bigcap_{i=1}^{p} \varrho_i^{-1}(N_i)$ where ϱ_i is in N_* an N_i is a neighborhood of 0 in C, the complex numbers. By hypothesis $\varrho_i \circ \Phi_n$ is in M_* for each i and n. Thus for a fixed i the sequence $\{\varrho_j \circ \Phi_n\}$ satisfies the hypothesis of Theorem 2 and by part (c) of that theorem the sequence $\{\varrho_i \circ \Phi_n\}$ is equicontinuous at 0 for the topology $\sigma(M, M_*) | \mathfrak{I} \cap L$. Thus for the given neighborhood N_i there is a $\sigma(N, M_*) | \mathfrak{I} \cap L$ neighborhood A_i of 0 such that

$$\varrho_i(\Phi_n(A_i)) \subseteq N_i$$

for all n. Obtaining such a set A_i for each i, we obtain from a routine calculation that

$$\Phi_n(\bigcap_{i=1}^{p} A_i) \subseteq \bigcap_{i=1}^{p} \Phi_n(A_i) \subseteq \bigcap_{i=1}^{p} \varrho_i^{-1}(N_i)$$

for all n. Thus the collection $\{\psi_n\}$ is equicontinuous with respect to the topologies $\sigma(M, M_*) \,|\, \mathfrak{I} \cap L$ and $\sigma(N, N_*)$.

We now recall that the Φ_n are uniformly bounded on norm-bounded sets. Such a set is $\mathfrak{I} \cap L$ and so the maps $\{\psi_n\}$ map $\mathfrak{I} \cap L$ into a norm-bounded subset of N. From the general theory of VON NEUMANN algebras, for any VON NEUMANN algebra S the topology $\sigma(S, S_*)$ relativized to a norm-bounded subset of S is the same as the weak operator topology relativized to that subset; see ([4], p. 34). Thus $\{\psi_n\}$ is equicontinuous at 0 with respect to the weak operator topology on $\mathfrak{I} \cap L$ and N. This proves (b).

For part (c), let $\{P_\alpha\}$ be a downward directed commutative collection of projections in M with g.l.b. $\{P_\alpha\} = 0$. Let L be the commutative VON NEUMANN subalgebra of M generated by $\{P_\alpha\}$. We note that $P_\alpha \to 0$ in the weak operator topology. Let $\varepsilon > 0$ and x in $\mathcal{H}$. Consider the weak operator neighborhood $\{A \,||(Ax, x)| < \varepsilon\} = V$ of 0 in N. By part (*b*) there is a weak operator neighborhood U of 0 in M such that $\psi_n(U) \subseteq V$ for all n. Noting that there is an α_0 such that P_α is in U for all $\alpha \geqslant \alpha_0$, the conclusion then follows.

Finally for part (d) it is easily seen that Φ is positive if each Φ_n is. To show that Φ is normal, let $\{A_\alpha\}$ be an increasing family of positive operators in M with l.u.b. $\{A_\alpha\} = A$. We note first that by hypothesis, if ϱ is in N_*, then $\varrho \circ \Phi_n$ is a normal linear functional and by part (b) of Theorem 2, $\varrho \circ \Phi$ is normal. Since each $\varrho \circ \Phi_n$ is positive if ϱ is positive, then $\varrho \circ \Phi$ is positive. Then by the proof of Theorem 2, p. 53 of [4], Φ will be continuous on norm-bounded subsets of M in the weak operator topologies on M and N.

Now the set $\{A_\alpha\} \cup \{A\}$ is a norm-bounded subset of M. It is also true that A, being the l.u.b. $\{A_\alpha\}$, is in the weak operator closure of $\{A_\alpha\}$ ([4], p. 321). Thus from the previous paragraph $\Phi(A)$ is in the weak operator closure of $\{\Phi(A_\alpha)\}$, and in addition, $\Phi(A) \leqslant \Phi(A)$ for all α since Φ is positive. Since $\Phi(A)$ is an upper bound of $\{\Phi(A_\alpha)\}$ and a weak operator limit point of the same collection, then $\Phi(A)$ is the l.u.b. of $\{\Phi(A_\alpha)\}$ ([4], p. 321). Thus $\Phi(A) = \Phi$ (l.u.b. $\{A_\alpha\}) =$ l.u.b. $\{\Phi(A_\alpha)\}$. So Φ is normal and consequently also completely additive. ∎

We now apply Theorem 4 to the case when $\{\Phi_n\}$ is a collection of expectation mappings.

III. - Expectation mappings.

For this section, we assume that $N \subseteq M$ and that N contains the identity map I.

The first part of the following theorem states that if each Φ_n in Theorem 4 is an expectation mapping, then so is Φ.

The second part is analogous to (a) of the classical VITALI-HAHN-SAKS theorem stated in the introduction.

Theorem 5. *Suppose the set $\{\Phi_n\}$ is as in Theorem 4 and in addition suppose that each Φ_n is an expectation mapping of M on N. Then Φ is also an expectation mapping of M on N.*

Let ψ be a continuous expectation mapping of M on N with the property: whenever A is a Hermitian element with $\psi(A)=0$, then $\Phi_n(A)=0$ for all n. Then for every $\varepsilon>0$, there is a $\delta>0$ such that for any Hermitian A in M

$$\|\psi(A)\|<\delta \qquad \textit{implies} \qquad \|\Phi_n(A)\|<\varepsilon$$

for all n.

Proof. The fact that Φ is adjoint-preserving, positive, and satisfies $\Phi(I)=I$ follows from the observation that each Φ_n has these properties.

To show that $A\Phi(B)=\Phi(AB)$ for every A in N and B in M, it is sufficient to note that

$$\begin{aligned}\|A\Phi(B)-\Phi(AB)\| &= \|A\Phi(B)-A\Phi_n(B)-\Phi_n(AB)-\Phi(AB)\| \\ &\leqslant \|A\|\,\|\Phi(B)-\Phi_n(B)\|+\|\Phi_n(AB)-\Phi(AB)\|\to 0\,.\end{aligned}$$

Thus Φ is an expectation mapping from M into N.

The proof of the second assertion in the Theorem is modeled after the proof of the classical VITALI-HAHN-SAKS theorem; see, for example, A. ZAANEN ([7], p. 332, theorem 4). Suppose that the assertion is false. Then there is an $\varepsilon_0>0$ such that, for every positive integer p, there is a Hermitian element A_p in M and a positive integer n_p such that

$$\|\psi(A_p)\|<\frac{1}{p} \qquad \text{and} \qquad \|\Phi_{n_p}(A_p)\|\geqslant\varepsilon_0\,.$$

We have that

$$\Phi_{n_p}(A_p) = \Phi_{n_p}[\psi(A_p) - \psi(A_p) + A_p].$$

But $\psi(A_p - \psi(A_p)) = 0$ and $A_p - \psi(A_p)$ is Hermitian since A_p is Hermitian and ψ is adjoint-preserving. Thus by hypothesis $\Phi_{n_p}[A_p - \psi(A_p)] = 0$. Thus $\Phi_{n_p}(A_p) = \Phi_{n_p}(\psi(A_p))$. Since the Φ'_{n_p}s are uniformly bounded in norm by some $K > 0$, we have

$$\varepsilon_0 \leqslant \|\Phi_{n_p}(A_p)\| = \|\Phi_{n_p}(\psi(A_p))\| \leqslant K\|\psi(A_p)\|,$$

for all p. But $\|\psi(A_p)\| \to 0$ as $p \to \infty$, a contradiction. ■

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