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B. B. SINHA and G. DAS (*)

General F-Connection and H-Projectively Related Connections in Almost Product Space. (**)

In this paper we have studied general F-connection and connections which are H-projectively related to each other in an almost product space.

1. - Introduction.

Let us consider an n-dimensional manifold M_n of differentiability class C^{r+1}. Let there be defined in M_n a vector valued linear function F such that

$$F(X) = \overline{X}\,, \tag{1.1}$$

where X is a vector field. If it satisfies

$$\overline{\overline{X}} - X = 0 \tag{1.2}$$

and there is given a positive definite Riemannian metric tensor g, such that

$$g(\overline{X},\, \overline{Y}) = g(X,\, Y)\,, \tag{1.3}$$

the M_n is called an almost product space.

(*) Indirizzo: Department of Mathematics, Banaras Hindu University, Varanasi-5, India.

(**) Ricevuto: 10-III-1970.

A connection D is said to be an F-connection if

$$(D_X F)(Y) = 0 . \tag{1.4}$$

The torsion tensor S and s of two connections D and E are related by

$$S(X, Y) - s(X, Y) = D_X Y - D_Y X - E_X Y + E_Y X . \tag{1.5}$$

An F-connection D is called half-symmetric if the torsion tensor S satisfies

$$S(X, Y) + S(\overline{X}, \overline{Y}) = -\overline{S(X, \overline{Y})} - \overline{S(\overline{X}, Y)} , \tag{1.6a}$$

$$S(X, \overline{Y}) + S(\overline{X}, Y) = -\overline{S(X, Y)} - \overline{S(\overline{X}, \overline{Y})} . \tag{1.6b}$$

The F-connection D is called semi-symmetric if

$$n S(X, Y) = T(Y)X - T(X)Y - T(\overline{Y})\overline{X} + T(\overline{X})\overline{Y} , \tag{1.7}$$

where

$$T(X) = (C_1^1 S)(X) . \tag{1.8}$$

NIJENHUIS tensor N of the connection E in the almost product space M_n is given by

$$\begin{cases} N(X, Y) = [\overline{X}, \overline{Y}] + [X, Y] - \overline{[X, \overline{Y}]} - \overline{[\overline{X}, Y]} \\ \qquad = E_{\overline{X}}\overline{Y} - E_{\overline{Y}}\overline{X} + E_X Y - E_Y X - \overline{E_X \overline{Y}} + \overline{E_{\overline{Y}} X} - \overline{E_{\overline{X}} \overline{Y}} + \overline{E_Y \overline{X}} . \end{cases} \tag{1.9}$$

It can be verified that

$$N(X, Y) = N(\overline{X}, \overline{Y}) = -\overline{N(\overline{X}, Y)} = -\overline{N(X, \overline{Y})} \tag{1.10a}$$

and consequently

$$N(X, \overline{Y}) = N(\overline{X}, Y) = -\overline{N(\overline{X}, \overline{Y})} = -\overline{N(X, Y)} . \tag{1.10b}$$

2. - F-connection.

Theorem 2.1. *Let E be an arbitrary connection, then the connection D defined by*

$$(2.1a)\quad \begin{cases} D_X Y = \alpha(E_X Y + \overline{E_X \overline{Y}}) + \beta(E_X \overline{Y} + \overline{E_X Y}) + \\ \qquad + \gamma(E_{\overline{X}} Y + \overline{E_{\overline{X}} \overline{Y}}) + \delta(E_{\overline{X}} \overline{Y} + \overline{E_{\overline{X}} Y}) , \end{cases}$$

is the most general F-connection of this type.

Proof. Let us put

$$(2.2)\quad \begin{cases} D_X Y = \alpha E_X Y + \beta E_X \overline{Y} + \gamma E_{\overline{X}} Y + \delta E_{\overline{X}} \overline{Y} + \\ \qquad + \theta \overline{E_X Y} + \varphi\, \overline{E_X \overline{Y}} + \varrho\, \overline{E_{\overline{X}} Y} + \sigma\, \overline{E_{\overline{X}} \overline{Y}} . \end{cases}$$

Barring this equation throughout, we get

$$(2.3)\quad \begin{cases} \overline{D_X Y} = \alpha\, \overline{E_X Y} + \beta\, \overline{E_X \overline{Y}} + \gamma\, \overline{E_{\overline{X}} Y} + \delta\, \overline{E_{\overline{X}} \overline{Y}} + \\ \qquad + \theta E_X Y + \varphi E_X \overline{Y} + \varrho E_{\overline{X}} Y + \sigma E_{\overline{X}} \overline{Y} . \end{cases}$$

Also barring Y in (2.2), using (1.2) and (1.4), we get

$$(2.4)\quad \begin{cases} \overline{D_X Y} = \alpha E_X \overline{Y} + \beta E_X Y + \gamma E_{\overline{X}} \overline{Y} + \delta E_{\overline{X}} Y + \theta E_X Y + \varphi E_X Y + \\ \qquad + \varrho E_{\overline{X}} \overline{Y} + \sigma E_{\overline{X}} Y + \theta E_X Y + \varphi E_X \overline{Y} + \varrho E_{\overline{X}} Y + \sigma E_{\overline{X}} \overline{Y} . \end{cases}$$

Comparing (2.3) and (2.4), we get

$$\varphi = \alpha , \qquad \beta = \theta , \qquad \gamma = \sigma , \qquad \delta = \varrho$$

which on substitution in (2.2) give (2.1a).

Corollary 2.1. *The equation* (2.1a) *is equivalent to*

$$(2.1b)\quad \begin{cases} D_X \overline{Y} = \alpha(E_X \overline{Y} + \overline{E_X Y}) + \beta(E_X Y + \overline{E_X \overline{\overline{Y}}}) + \\ \qquad + \gamma(E_{\overline{X}} \overline{Y} + \overline{E_{\overline{X}} Y}) + \delta(\overline{E_X \overline{\overline{Y}}} + E_X Y) , \end{cases}$$

$$\text{(2.1c)}\quad \left\{\begin{aligned} D_{\bar{X}}Y = {} & \alpha(E_{\bar{X}}Y + \overline{E_{\bar{X}}\bar{Y}}) + \beta(E_{\bar{X}}\bar{Y} + \overline{E_{\bar{X}}Y}) + \\ & + \gamma(E_XY + \overline{E_X\bar{Y}}) + \delta(E_X\bar{Y} + \overline{E_XY}), \end{aligned}\right.$$

$$\text{(2.1d)}\quad \left\{\begin{aligned} D_{\bar{X}}\bar{Y} = {} & \alpha(E_{\bar{X}}\bar{Y} + \overline{E_{\bar{X}}Y}) + \beta(E_{\bar{X}}Y + \overline{E_{\bar{X}}\bar{Y}}) + \\ & + \gamma(E_X\bar{Y} + \overline{E_XY}) + \delta(E_XY + \overline{E_X\bar{Y}}), \end{aligned}\right.$$

$$\text{(2.1e)}\quad \left\{\begin{aligned} \overline{D_XY} = {} & \alpha(\overline{E_XY} + E_X\bar{Y}) + \beta(\overline{E_X\bar{Y}} + E_XY) + \\ & + \gamma(\overline{E_{\bar{X}}Y} + E_{\bar{X}}\bar{Y}) + \delta(E_{\bar{X}}Y + \overline{E_{\bar{X}}\bar{Y}}), \end{aligned}\right.$$

$$\text{(2.1f)}\quad \left\{\begin{aligned} \overline{D_X\bar{Y}} = {} & \alpha(\overline{E_X\bar{Y}} + E_XY) + \beta(\overline{E_XY} + E_X\bar{Y}) + \\ & + \gamma(\overline{E_{\bar{X}}\bar{Y}} + E_{\bar{X}}Y) + \delta(E_{\bar{X}}\bar{Y} + \overline{E_{\bar{X}}Y}), \end{aligned}\right.$$

$$\text{(2.1g)}\quad \left\{\begin{aligned} \overline{D_{\bar{X}}Y} = {} & \alpha(\overline{E_{\bar{X}}Y} + E_{\bar{X}}\bar{Y}) + \beta(\overline{E_{\bar{X}}\bar{Y}} + E_{\bar{X}}Y) + \\ & + \gamma(\overline{E_XY} + E_X\bar{Y}) + \delta(\overline{E_X\bar{Y}} + E_{\bar{X}}\bar{Y}), \end{aligned}\right.$$

$$\text{(2.1h)}\quad \left\{\begin{aligned} \overline{D_{\bar{X}}\bar{Y}} = {} & \alpha(\overline{E_{\bar{X}}\bar{Y}} + E_{\bar{X}}Y) + \beta(\overline{E_{\bar{X}}Y} + E_{\bar{X}}\bar{Y}) + \\ & + \gamma(\overline{E_X\bar{Y}} + E_XY) + \delta(\overline{E_XY} + E_X\bar{Y}). \end{aligned}\right.$$

Proof. Barring different vectors in (2.1a), we obtain (2.1b) to (2.1h) with the help of (1.1) and (1.2).

Theorem 2.2. *Let S be the torsion tensor of D. Let*

$$\text{(2.5)}\qquad [X, Y] = E_XY - E_YX,$$

then

$$\text{(2.6a)}\quad \left\{\begin{aligned} & S(X, Y) + S(\bar{X}, \bar{Y}) = \\ & = (\alpha - 1)([X, Y] + [\bar{X}, \bar{Y}]) + \alpha(\overline{[X, \bar{Y}]} + \overline{[\bar{X}, Y]}) + (\beta + \gamma)([X, \bar{Y}] + \\ & + \overline{[\bar{X}, \bar{Y}]} + [\bar{X}, Y] + \overline{[X, Y]}) + \delta([\bar{X}, \bar{Y}] + [X, Y] + \overline{[\bar{X}, Y]} + \overline{[X, \bar{Y}]}), \end{aligned}\right.$$

$$
(2.6b)\quad \begin{cases} S(\bar X,\, Y) + S(X,\, \bar Y) = \\ = (\alpha-1)([\bar X,\, Y]+[X,\, \bar Y]) + \alpha(\overline{[\bar X,\, \bar Y]}+\overline{[X,\, Y]}) + (\beta+\gamma)([\bar X,\, \bar Y] + \\ + \overline{[X,\, \bar Y]}+[X,\, Y]+\overline{[\bar X,\, Y]}) + \delta([X,\, \bar Y]+[\bar X,\, Y]+\overline{[X,\, Y]}+\overline{[\bar X,\, \bar Y]}) , \end{cases}
$$

$$
(2.6c)\quad \begin{cases} \overline{S(X,\, Y)} + \overline{S(\bar X,\, \bar Y)} = \\ = (\alpha-1)(\overline{[X,\, Y]}+\overline{[\bar X,\, \bar Y]}) + \alpha([X,\, \bar Y]+[\bar X,\, Y]) + (\beta+\gamma)(\overline{[X,\, \bar Y]} + \\ +[\bar X,\, \bar Y]+\overline{[\bar X,\, Y]}+[X,\, Y]) + \delta(\overline{[\bar X,\, \bar Y]}+\overline{[X,\, Y]}+[\bar X,\, Y]+[X,\, \bar Y]) , \end{cases}
$$

$$
(2.6d)\quad \begin{cases} \overline{S(\bar X,\, Y)} + \overline{S(X,\, \bar Y)} = \\ = (\alpha-1)(\overline{[\bar X,\, Y]}+\overline{[X,\, \bar Y]}) + \alpha([\bar X,\, \bar Y]+[X,\, Y]) + (\beta+\gamma)(\overline{[\bar X,\, \bar Y]} + \\ +[X,\, \bar Y]+\overline{[X,\, Y]}+\overline{[\bar X,\, Y]}) + \delta(\overline{[X,\, \bar Y]}+\overline{[\bar X,\, Y]}+[X,\, Y]+[\bar X,\, \bar Y]) . \end{cases}
$$

Proof. Interchanging X, Y in (2.1a) and (2.1d), subtracting the resulting equations from the sum of (2.1a) and (2.1d), we obtain (2.6a) with the help of (1.3). (2.6b), (2.6c) and (2.6d) can be proved by barring different vectors in (2.6a) and using (1.2).

Corollary 2.2. *The equation* (2.6) *can be written as*

$$
(2.7a)\quad \begin{cases} S(X,\, Y) + S(\bar X,\, \bar Y) = \\ = (\alpha-1)(N(X,\, Y) + \overline{[\bar X,\, Y]} + \overline{[X,\, \bar Y]}) + \alpha(\overline{[X,\, \bar Y]} + \overline{[\bar X,\, Y]}) + \\ + (\beta+\gamma) + (2\,[\bar X,\, Y] + 2\,[X,\, \bar Y] - N(X,\, \bar Y)) + \\ + \delta(N(X,\, Y) + 2\,\overline{[X,\, \bar Y]} + 2\,\overline{[\bar X,\, Y]}) , \end{cases}
$$

$$
(2.7b)\quad \begin{cases} S(X,\, \bar Y) + S(\bar X,\, Y) = \\ = (\alpha-1)(N(X,\, \bar Y) + \overline{[\bar X,\, \bar Y]} + \overline{[X,\, Y]}) + \alpha(\overline{[X,\, Y]} + \overline{[\bar X,\, \bar Y]}) + \\ + (\beta+\gamma)(2\,[\bar X,\, \bar Y] + 2\,[X,\, Y] - N(X,\, Y)) + \\ + \delta(N(X,\, \bar Y) + 2\,\overline{[X,\, Y]} + 2\,\overline{[\bar X,\, \bar Y]}) , \end{cases}
$$

(2.7c)
$$\begin{cases} \overline{S(X,\, Y)} + \overline{S(\bar{X},\, \bar{Y})} = \\ = (\alpha - 1)([\bar{X},\, Y] + [X,\, \bar{Y}] - N(X,\, \bar{Y})) + \alpha([X,\, \bar{Y}] + [\bar{X},\, Y]) + \\ + (\beta + \gamma)(2\,\overline{[\bar{X},\, Y]} + 2\,\overline{[X,\, \bar{Y}]} + N(X,\, Y)) + \\ + \delta(2\,[\bar{X},\, Y] + 2\,[X,\, \bar{Y}] - N(X,\, \bar{Y}))\,, \end{cases}$$

(2.7d)
$$\begin{cases} \overline{S(X,\, \bar{Y})} + \overline{S(\bar{X},\, Y)} = \\ = (\alpha - 1)([\bar{X},\, \bar{Y}] + [X,\, Y] - N(X,\, Y)) + \alpha([X,\, Y] + [\bar{X},\, \bar{Y}]) + \\ + (\beta + \gamma)(2\,\overline{[\bar{X},\, \bar{Y}]} + 2\,\overline{[X,\, Y]} + N(X,\, \bar{Y})) + \\ + \delta(2\,[X,\, Y] + 2\,[\bar{X},\, \bar{Y}] - N(X,\, Y))\,. \end{cases}$$

Proof. Using (1.10) in (2.6), we obtain (2.7).

Corollary 2.3. *Let us put*

(2.8a) $$N(X,\, Y) = 2\,[\bar{X},\, \bar{Y}] + 2[X,\, Y] = -2\,\overline{[\bar{X},\, Y]} - 2\,\overline{[X,\, \bar{Y}]}$$

or

(2.8b) $$N(X,\, \bar{Y}) = 2\,[\bar{X},\, Y] + 2\,[X,\, \bar{Y}] = -2\,\overline{[\bar{X},\, \bar{Y}]} - 2\,\overline{[X,\, Y]}\,,$$

then the above equations (2.7) *assume the forms*

(2.9a) $$S(X,\, Y) + S(\bar{X},\, \bar{Y}) = \tfrac{1}{2} N(X,\, Y)\,,$$

(2.9b) $$S(X,\, \bar{Y}) + S(\bar{X},\, Y) = \tfrac{1}{2} N(\bar{X},\, Y)\,,$$

(2.9c) $$\overline{S(X,\, Y)} + \overline{S(\bar{X},\, \bar{Y})} = -\tfrac{1}{2} N(X,\, \bar{Y})$$

and

(2.9d) $$\overline{S(X,\, \bar{Y})} + \overline{S(\bar{X},\, Y)} = -\tfrac{1}{2} N(X,\, Y)$$

whence

Corollary 2.4.

(2.10a) $$S(X,\, Y) + S(\bar{X},\, \bar{Y}) + \overline{S(X,\, \bar{Y})} + \overline{S(\bar{X},\, Y)} = 0\,,$$

(2.10b) $$S(X,\, \bar{Y}) + S(\bar{X},\, Y) + \overline{S(X,\, Y)} + \overline{S(\bar{X},\, \bar{Y})} = 0\,.$$

Proof. Substituting from (2.8) in (2.7), we obtain (2.9). Eliminating N between (2.9), we obtain (2.10), which shows that the F-connection D is half-symmetric.

3. - H-projectively related connections.

Let L be the unit tanget vector field to a given curve C in an almost product manifold M_n. If the curve satisfies the differential equation

$$D_L L = aL + b\overline{L} \tag{3.1}$$

where a, b are functions defined along the curve C, we call C a holomorphically plane curve. If the two F-connections D and E have all holomorphically plane curves in common, we say that they are H-projectively related to each other.

Theorem 3.1. *Let D and E be two semi-symmetric F-connections. These are H-projectively related to each other if and only if*

$$D_X Y = E_X Y + P(Y)X + Q(X)Y + P(\overline{Y})\overline{X} + R(X)\overline{Y} \tag{3.2}$$

for certain vector fields P, Q and R.

Proof. Suppose (3.2) is satisfied. Then we have

$$D_L L = E_L L + \{P(L) + Q(L)\}\, L + \{P(\overline{L}) + R(L)\}\, \overline{L}\, ,$$

which shows that D and E have all holomorphically plane curves in common.

Conversly, suppose D and E have all holomorphically plane curves in common. Let us put

$$A(X, Y) = D_X Y - E_X Y\, , \tag{3.3}$$

then we must have

$$A(L, L) = \alpha L + \beta\overline{L}\, . \tag{3.4}$$

Since D and E are F-connections, we have, by barring Y and A in (3.3) and making use of (1.4),

$$\overline{A(X, \overline{Y})} = \overline{D_X\overline{Y}} - \overline{E_X\overline{Y}} = D_X Y - E_X Y\, . \tag{3.5}$$

Consequently from (3.3) and (3.5), we have

$$A(X, Y) - \overline{A(\overline{X}, \overline{Y})} = 0 . \tag{3.6}$$

(3.4) suggest that the most general form of $A(X, Y)$ is given by

$$\left\{\begin{aligned} 2A(X, Y) &= \{U(Y) + W(\overline{Y})\}X + \{U(X) + W(\overline{X})\}Y + \\ &+ \{V(Y) + G(\overline{Y})\}\overline{X} + \{V(X) + G(\overline{X})\}\overline{Y} + 2P(X, Y)) , \end{aligned}\right. \tag{3.7}$$

where

$$P(X, Y) = S(X, Y) - s(X, Y) , \tag{3.8}$$

S and s being the torsion tensors of D and E respectively.
Let us put

$$M(X) \overset{\text{def}}{=} T(X) - t(X) \tag{3.9}$$

then

$$nP(X, Y) = M(Y)X - M(X)Y - M(\overline{Y})\overline{X} + M(\overline{X})\overline{Y} . \tag{3.10}$$

Substituting from (3.10) in (3.7) we obtain

$$\left\{\begin{aligned} A(X, Y) = &\left\{U(Y) + W(\overline{Y}) + \frac{2}{n} M(Y)\right\} X + \\ &+ \left\{U(X) + W(\overline{X}) - \frac{2}{n} M(X)\right\} Y + \\ &+ \left\{V(Y) + G(\overline{Y}) - \frac{2}{n} M(\overline{Y})\right\} \overline{X} + \left\{V(X) + G(\overline{X}) + \frac{2}{n} M(\overline{X})\right\} \overline{Y} . \end{aligned}\right. \tag{3.11a}$$

Consequently

$$\left\{\begin{aligned} \overline{2A(\overline{X}, \overline{Y})} = &\left\{U(\overline{Y}) + W(Y) + \frac{2}{n} M(\overline{Y})\right\} \overline{X} + \\ &+ \left\{U(X) + W(\overline{X}) - \frac{2}{n} M(X)\right\} Y + \\ &+ \left\{V(\overline{Y}) + G(Y) - \frac{2}{n} M(Y)\right\} X + \left\{V(X) + G(\overline{X}) + \frac{2}{n} M(\overline{X})\right\} \overline{Y} . \end{aligned}\right. \tag{3.11b}$$

Subtracting (3.11b) from (3.11a) and using (3.6), we get

$$\left\{U(Y) + W(\bar{Y}) - G(Y) - V(\bar{Y}) + \frac{4}{n} M(Y)\right\} X +$$

$$+ \left\{V(Y) + G(\bar{Y}) - U(\bar{Y}) - W(Y) - \frac{4}{n} M(\bar{Y})\right\} \bar{X} = 0 .$$

Since this equation holds for arbitrary X, we have

$$\frac{4}{n} M(Y) = V(\bar{Y}) + G(Y) - U(Y) - W(\bar{Y}) \tag{3.12a}$$

and

$$\frac{4}{n} M(\bar{Y}) = V(Y) + G(\bar{Y}) - U(\bar{Y}) - W(Y) . \tag{3.12b}$$

Substituting from (3.12) in (3.11a), we obtain

$$\begin{cases} 2A(X, Y) = \{U(Y) + W(\bar{Y}) + V(\bar{Y}) + G(Y)\} \dfrac{X}{2} + \\ \quad + \{V(Y) + G(\bar{Y}) + U(\bar{Y}) + W(Y)\} \dfrac{\bar{X}}{2} + \\ \quad\quad + \{3U(X) + 3W(\bar{X}) - V(\bar{X}) - G(X)\} \dfrac{Y}{2} + \\ \quad + \{3V(X) + 3G(\bar{X}) - U(\bar{X}) - W(X)\} \dfrac{\bar{Y}}{2} . \end{cases} \tag{3.13}$$

Putting

$$P(Y) = \tfrac{1}{4}\{U(Y) + W(\bar{Y}) + V(\bar{Y}) + G(Y)\} \tag{3.14a}$$

$$Q(X) = \tfrac{1}{4}\{3U(X) + 3W(\bar{X}) - V(\bar{X}) - G(X)\} \tag{3.14b}$$

and

$$R(X) = \tfrac{1}{4}\{3V(X) + 3G(\bar{X}) - U(\bar{X}) - W(X)\} \tag{3.14c}$$

in (3.13), we obtain (3.2) with the help of (3.3).

Corollary 3.1. *Two symmetric F-connections D and E are H-projectively related to each other if and only if*

$$D_X Y = E_X Y + P(Y)X + P(X)Y + P(\overline{Y})\overline{X} + P(\overline{X})\overline{Y} . \tag{3.15}$$

Proof. When the connections are symmetric,

$$M(Y) = 0 .$$

Consequently from (3.12a)

$$V(\overline{Y}) + G(Y) = U(Y) + W(\overline{Y}) .$$

Hence (3.14) become

$$P(Y) = \{U(Y) + W(\overline{Y})\} = Q(\overline{Y})$$

and

$$R(Y) = P(\overline{Y}) .$$

Substituting from these equations in (3.2), we obtain (3.15).

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