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On a Relation Between Harmonic Summability and Riemann Summability. (**)

1. - Definitions and notations.

Let $\sum a_n$ be a given infinite series with s_n as its n-th partial sum. We denote by S_n^k the n-th CESÀRO sum of order k of this series. An infinite series $\sum a_n$ is said to be (R, 1) summable to the sum s, if the series

$$(1.1)\qquad \sum_{n=1}^{\infty} a_n \frac{\sin nt}{nt}$$

converges in some interval $0 < t < t_0$ and if

$$(1.2)\qquad \lim_{t\to 0} \sum_{n=1}^{\infty} a_n \frac{\sin nt}{nt} = s .$$

The summability (R, 1) is sometimes referred to as LEBESGUE summability.

The series $\sum a_n$ is said to be (R_1) summable to the sum s, if the series

$$(1.3)\qquad \frac{2}{\pi} \sum_{n=1}^{\infty} s_n \frac{\sin nt}{n}$$

converges in some interval $0 < t < t_0$ and if,

$$(1.4)\qquad \lim_{t\to 0} t \frac{2}{\pi} \sum_{n=1}^{\infty} s_n \frac{\sin nt}{nt} = s .$$

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It is said to be (C, K)-summable to s, if

$$\lim_{n\to\infty} S_n^k/A_n^k = s \qquad \text{where} \qquad A_n^k = \binom{n+k}{n} \qquad (K > -1) \tag{1.5}$$

and harmonic summable to the sum s, if $t_n \to s$ as $n \to \infty$ where,

$$t_n = \frac{T_n}{\log(n+1)} = \frac{1}{\log(n+1)} \sum_{v=1}^{n} p_{n-v} s_v\,, \qquad p_n = \frac{1}{n+1}\,; \quad T_0 = 0. \tag{1.6}$$

It is well known that summability (R, 1) and (R_1) are not comparable [**1**]. Concerning (R, 1) summability and (R_1) summability SZÀSZ ([**3**], [**4**]) has proved the following

Theorem A. If $\sum a_n$ is $(C, 1-\alpha)$ summable for some positive $\alpha < 1$, and if

$$\sigma_n = \sum_{v=1}^{n} |S_v^{-\alpha}| = O(n^{1-\alpha}) \qquad \text{as } n \to \infty, \tag{1.7}$$

then the series $\sum a_n$ is summable by (R, 1) and (R_1) methods.

Recently VARSHNEY [**5**] has proved an analogous theorem for harmonic summability. His result is as follows:

Theorem B. If $\sum_1^\infty a_n$ is harmonic summable and if

$$\sum_{v=1}^{n} |T_v - T_{v-1}| = O(\log n) \qquad \text{as } n \to \infty, \tag{1.8}$$

then $\sum_1^\infty a_n$ is LEBESGUE summable.

Quite recently author [**6**] has proved the following theorem for (R_1) summability.

Theorem C. If $\sum a_n$ is harmonic summable and if

$$\sum_1^n |T_v| = O(\log n) \qquad (n \to \infty) \tag{1.9}$$

then $\sum_{n=1}^\infty a_n$ is (R_1) summable.

The question arises as to whether the condition (1.9) can be replaced by a lighter condition (1.8).

The object of this Note is to answer this question in affirmative.
In what follows we shall prove the following

Theorem. *If* $\sum_{1}^{\infty} a_n$ *is harmonic summable and if,*

((1.10) $$W_n = \sum_{v=1}^{n} |T_v - T_{v-1}| = O(\log n) \qquad \text{as } n \to \infty$$

then $\sum_{1}^{\infty} a_n$ *is* (R_1) *summable.*

2. - We set

$$\Big(\sum_{n=0}^{\infty} p_n x^n\Big)^{-1} = \sum_{n=0}^{\infty} c_n x^n$$

then we have

(2.1) $$a_n = \sum_{v=1}^{n} c_{n-v}(T_v - T_{v-1})$$

and

(2.2) $$s_n = \sum_{v=1}^{n} c_{n-v} T_v .$$

It is well known that [2] $c_n = O\big(1/(n \log^2 n)\big)$.

(2.3) $$d_n = \sum_{v=0}^{n} c_v = O\left(\frac{1}{\log n}\right) .$$

We may assume without loss of generality that

$$T_n = o(\log n) \qquad \text{as } n \to \infty.$$

3. - We require the following lemmas for the proof of our theorem.

Lemma 1 [5]. Let $K_n = \sum_{v=n}^{\infty} b_v/v$, where $b_v = T_v - T_{v-1}$ and $\sum_{1}^{n} |T_v - T_{v-1}| = O(\log n)$, then $K_n = o(\log n/n)$. Also $K'_n = \sum^{n} K_v = o(\log n)$.

Lemma 2 [5]. Let

$$\eta_v(t) = \sum_{n=0}^{\infty} c_n \frac{\sin (n+v)t}{n+v}$$

then

$$(3.1) \qquad \eta_v(t) = O\left(\frac{1}{v \log T}\right), \qquad T = \left[\frac{1}{t}\right], \qquad (0 < t < 1), \ (v \geqslant 1).$$

Lemma 3 [4]. If the series $\sum_{n=1}^{\infty} s_n (\sin nt/n)$ converges in $0 < t < t_0$ then;

$$(3.2) \qquad \sum_{n=1}^{\infty} s_n \frac{\sin nt}{n} = \sum_{1}^{\infty} a_n \varrho_n(t)$$

where

$$(3.3) \qquad \varrho_n(t) = \sum_{v=n}^{\infty} \frac{\sin vt}{v}.$$

Conversely, if $s_n/n \to 0$, then the convergence of $\sum_{1}^{\infty} a_n \varrho_n(t)$ implies (3.2).

Lemma 4. *If* $\beta_v(t) = \sum_{n=0}^{\infty} c_n \varrho_{n+v}(t)$, *then*

$$(3.4) \qquad \beta_v(t) = O(1/v\, t \log T), \qquad v \geqslant 1 \ \textit{where} \ T = [1/t].$$

Proof. Let $\beta_v(t) = (\sum_{n=0}^{T} + \sum_{n=T+1}^{\infty}) c_n \varrho_{n+v}(t) = U_1 + U_2$, say then,

$$U_2 = O\left(\sum_{n=T+1}^{\infty} \frac{1}{n \log^2 n} \cdot \frac{1}{(n+v)t}\right)$$

$$= O\left(\frac{1}{(v+T+1)t} \sum_{n=T+1}^{\infty} \frac{1}{n \log^2 n}\right) = \left(\frac{1}{v\, t \log T}\right),$$

$$U_1 = \sum_{n=0}^{T} c_n \varrho_{n+v}(t).$$

Applying ABELL's transformation to U_1 we have

$$U_1 = \sum_{n=0}^{T-1} d_n \Delta\, \varrho_{n+v}(t) + d_T\, \varrho_{T+v}(t) =$$

$$= O\left(\sum_{n=0}^{T-1} \frac{1}{\log(n+1)} \cdot \frac{1}{(n+v)}\right) + O\left(\frac{1}{\log T} \cdot \frac{1}{(v+T)t}\right) = O\left(\frac{1}{vt \log T}\right),$$

hence $U_1 + U_2 = O(1/(vt \log T))$.

Lemma 5.

$$\Delta\beta_v(t) = O\left(\frac{1}{v \log T}\right) \text{ [5]} \quad \textit{and} \quad \Delta^2\beta_v(t) = O\left(\frac{t}{v \log T}\right) \text{ [6]}.$$

4. - Proof of the theorem.

Since

$$\frac{s_n}{n} = \frac{1}{n} \sum_{v=0}^{n} c_{n-v}\, T_v$$

$$= \frac{1}{n} \sum_{v=0}^{n} O\left(\frac{1}{(n-v)\, \log^2 (n-v)}\right) o(\log v) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{o(\log n + k)}{k \log^2 k} =$$

$$= \frac{1}{n} \sum_{k=2}^{k-1} o\left(\frac{1}{k \log k}\right) = o\left(\frac{1}{n} \log\log n\right) = o(1),$$

therefore by virtue of Lemma 3 it is sufficient to prove that $\sum_{n=1}^{\infty} a_n \varrho_n(t)$ converges and its limit as $t \to 0$ is equal to zero. Using (2.1) we have

$$\text{(4.1)} \quad \begin{cases} \displaystyle\sum_{n=1}^{\infty} a_n \varrho_n(t) = \sum_{n=1}^{\infty} \varrho_n(t) \sum_{v=1}^{n} c_{n-v}\, (T_v - T_{v-1}) \\ \displaystyle\qquad = \sum_{v=1}^{\infty} (T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v}\, \varrho_n(t)\,. \end{cases}$$

The change of order of summation is justified for

$$\sum_{v=1}^{\infty} |T_v - T_{v-1}| \sum_{n=0}^{\infty} |c_n\, \varrho_{n+v}(t)| = O(\sum_{v=1}^{\infty} |T_v - T_{v-1}| 1/vt \sum_{n=0}^{\infty} |c_n|)$$

$$= O(\sum_{v=1}^{\infty} |T_v - T_{v-1}| 1/v) \qquad \text{(for fixed positive } t\text{)}$$

$$= O\left(\sum_{v=1}^{n-1} \frac{W_v}{v(v+1)} + \frac{W_n}{n}\right) = O\left(\sum_{v=1}^{n-1} \frac{\log v}{v(v+1)}\right) + O\left(\frac{\log n}{n}\right) = O(1) .$$

Thus the series in (4.1) converge absolutely.

Let

$$F(t) = \sum_{v=1}^{\infty} (T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v} \varrho_n(t) .$$

Now we choose a positive number μ, put $n = [\mu/t]$ and write

$$\sum_{v=1}^{\infty} (T_v - T_{v-1}) \beta_v(t) = (\sum_{v=1}^{n} + \sum_{n+1}^{\infty})(T_v - T_{v-1}) \beta_v(t)$$

$$= V_1 + V_2 , \qquad \text{say.}$$

From (3.4) we have

$$V_2 = O\left(\frac{1}{t \log T} \sum_{n+1}^{\infty} \frac{1}{v} |T_v - T_{v-1}|\right) .$$

Using ABEL's transformation we have

$$V_2 = O\left\{\frac{1}{t \log T}\left(\sum_{n+1}^{\infty} \frac{W_v}{v(v+1)} - \frac{W_n}{n+1}\right)\right\}$$

$$= O\left(\frac{\log n}{nt \log T}\right) + O\left(\frac{1}{t \log T} \sum_{n+1}^{\infty} \frac{\log v}{v(v+1)}\right)$$

$$= O\left(\frac{\log n}{nt \log T}\right) = O\left(\frac{\log \mu}{\mu}\right) .$$

Furthermore

$$V_1 = \sum_{v=1}^{n} (T_v - T_{v-1}) \beta_v(t) =$$

$$= \sum_{v=1}^{n} (K_v - K_{v+1})\, v\, \beta_v(t) \qquad \text{(Lemma 1)}$$

$$= \sum_{1}^{n} K_v\{v\, \beta_v(t) - (v-1)\beta_{v-1}(t)\} - nK_{n+1}\beta_n(t)$$

$$= \sum_{1}^{n} K_v \left\{v \sum_{n=0}^{\infty} c_n\, \varrho_{n+v} - (v-1) \sum_{n=0}^{\infty} c_n\, \varrho_{n+v-1}\right\} - nK_{n+1}\beta_n(t)$$

$$= \sum_{1}^{n} K_v \sum_{n=0}^{\infty} c_n\, \varrho_{n+v-1} - \sum_{1}^{n} K_v \sum_{n=0}^{\infty} c_n\, v\, \Delta\, \varrho_{n+v-1} - nK_{n+1}\beta_n(t)$$

$$= \sum_{1}^{n} K_v \beta_{v-1}(t) + o\left(\sum_{1}^{n} K_v\, v \frac{1}{v \log T}\right) + \frac{o\,(\log n)}{nt \log T}$$

$$= \sum_{1}^{n} K_v\, \beta_{v-1}(t) + o\left(\frac{\log n}{\log T}\right) + o\left(\frac{\log n}{nt \log T}\right).$$

Applyng Abel's transformation twice, to the same sum we have

$$\sum_{1}^{n} K_v \beta_{v-1}(t) =$$

$$= \sum_{1}^{n-2} \sum_{m=1}^{v} K'_m\, \Delta^2 \beta_{v-1}(t) + \sum_{m=1}^{n-1} K'_m \Delta\beta_{n-2}(t) + \sum_{m=1}^{n} K_m \beta_{n-1}(t) =$$

$$= o\left(\sum_{1}^{n-2} v \log v \frac{t}{(v-1)\log T}\right) + o\left(\sum_{m=1}^{n-1} \log m \frac{1}{(n-2)\log T}\right) + o\left(\frac{\log n}{nt \log T}\right).$$

Therefore

$$V_1 = o\left(\frac{nt \log n}{\log T}\right) + o\left(\frac{\log n}{\log T}\right) + o\left(\frac{\log n}{nt \log T}\right)$$

$$= o(\mu \log \mu) + o(\log \mu) + o(\log \mu/\mu).$$

Hence

$$V_1 + V_2 = O(\log \mu/\mu) + o(\mu \log \mu) \qquad \text{as } t \to 0.$$

Consequently,

$$\limsup_{t\to 0}\, |F(t) \leqslant O(\log \mu/\mu),$$

μ being arbitrarily large, we get

$$\lim_{t\to 0} F(t) = 0 .$$

This completes the Proof of our Theorem.

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