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# Temperature Distribution in Poiseuille Flow Between Two Parallel Flat Plates. (**)

## 1. - Introduction.

1. – PAI (1956) has given the velocity and temperature distributions for the POISEUILLE flow. However, he has given the solution of the energy equation without considering the rate of heat generation per unit volume in the fluid other than viscous dissipation. BHATNAGAR and TIKEKAR (1966) have obtained the temperature distribution in a channel bounded by two coaxial circular cylinders. They assumed that the rate of heat generation per unit volume as a function of time but did not include the effects of viscous dissipation. PURHOIT (1967) has given the temperature distribution of a viscous incompressible fluid flowing between two parallel flat plates (COUETTE flow). S. N. DUBEY (1970) has obtained the temperature distribution in a channel bounded by two parallel flat plates when viscous incompressible fluid is flowing through it, effects of viscous dissipation being taken into account. He has taken the rate of heat generation per unit volume in the fluid as a linear function of time in first part and an exponentially decreasing function of time in the second part.

In the present paper we propose to obtain the temperature distribution when the rate of heat generation per unit volume varies as $(n/2)$-th power of time $(n = -1, 0, 1, 2, ...)$ in the first part. Our attempt, thus, is to arrive at a relatively more general result which will be applicable to a wider range of situation and which includes DUBEY's result as a particular case. In the

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second part the temperature distribution in the same channel is studied in a situation where the rate of heat generation per unit volume is a sinusoidal function of time, meaning thereby that heat is periodically exchanged with the fluid, the fluid therefore receives heat as often as it loses. Effects of viscous dissipation have also been taken into account.

The expression for the temperature distributions in both the parts are derived with the conditions that the plates situated at $y = \pm y_0$ (i) have zero initial temperatures, and (ii) are always being kept at zero temperatures.

The result obtained is in complete aggrement with similar results obtained by BALLABH (1969) who has obtained the expression for the velocity by using the method of superposability.

In the present discussion velocity distribution is steady while the temperature distribution is unsteady. The temperature distribution does not influence the flow field of an incompressible fluid with constant properties. We have assumed a fluid having these properties.

## 2. - Energy equation and its solution.

If we assume that the temperature $T$ of the liquid is independent of its axial position, then the energy equation (PAI, 1956) in the present case reduces to

$$\frac{\partial T}{\partial t} = \frac{1}{\varrho C_v}\frac{\partial Q}{\partial t} + k'\frac{\partial^2 T}{\partial y^2} + Cy^2, \tag{1}$$

where $\partial Q/\partial t$ is the rate of heat generation per unit volume in the fluid,

$$k' = \frac{k}{\varrho C_v} \qquad \text{and} \qquad C = \frac{4\mu u_m^2}{\varrho C_v y_0^4}$$

are constants ($u_m$ represents the maximum velocity in the channel). The last term in equation (1) is viscous dissipation and is not neglected in the present investigation.

### Part I

## 3. - Flow when $\frac{1}{\varrho C_v}\frac{\partial Q}{\partial t} = at^{n/2}$.

Let us take,

$$\frac{1}{\varrho C_v}\frac{\partial Q}{\partial t} = at^{n/2}. \tag{2}$$

With this assumption equation (1), then, becomes,

$$\frac{\partial T}{\partial t} = a t^{n/2} + k' \frac{\partial^2 T}{\partial y^2} + C y^2 . \tag{3}$$

Now let $\overline{T} = \int\limits_0^\infty \exp(-st)\, T \, dt$ be the LAPLACE transform of $T$ and let $T_0$ be the initial value of $T$.

Multiplying equation (2) by $\exp(-st)$ and integrating between the limits 0 and $\infty$, we have,

$$\frac{\partial^2 \overline{T}}{\partial y^2} - p^2 \overline{T} = -\frac{1}{k'}\left[T_0 + \frac{Cy^2}{s} + \frac{a\sqrt{1+(n/2)}}{s^{1+(n/2)}}\right] , \tag{4}$$

where $p^2 = s/k'$. Now let us find $T_0$.

Initially the rate of heat generation is zero and the temperature is steady in the channel. Hence $\partial T_0/\partial t = 0$ and we obtain

$$\frac{d^2 T_0}{dy^2} = -\frac{C}{k'} y^2 . \tag{5}$$

The boundary conditions are

$$T_0 = 0 , \quad \text{when} \quad y = -y_0 ,$$

$$T_0 = 0 , \quad \text{when} \quad y = y_0 .$$

The solution of equation (5) under these boundary conditions is

$$T_0 = \frac{C}{12k'} (y_0^4 - y^4) .$$

Substituting this value of $\overline{T}_0$ in (4), we get

$$\frac{\partial^2 \overline{T}}{\partial y^2} - p^2 \overline{T} = -\frac{1}{k'}\left[\frac{C}{12k'} (y_0^4 - y^4) + \frac{Cy^2}{s} + \frac{a\sqrt{1+(n/2)}}{s^{1+(n/2)}}\right] . \tag{6}$$

The boundary conditions for $T$ are,

$$\begin{cases} \overline{T} = 0 & \text{when} \quad y = -y_0 , \\ \overline{T} = 0 & \text{when} \quad y = y_0 . \end{cases} \tag{7}$$

The solution of equation (6) under these boundary conditions is

(8)
$$\left\{\begin{aligned}\bar{T} &= \frac{C}{12k'}\left(\frac{y_0^4 - y^4}{s}\right) + \frac{a\sqrt{1+(n/2)}}{s^{2+(n/2)}}\left\{1 - \frac{\operatorname{Cosh} py}{\operatorname{Cosh} py_0}\right\} = \\ &= \frac{C}{12k'}\left(\frac{y_0^4 - y^4}{s}\right) + \frac{a\sqrt{1+(n/2)}}{s^{2+(n/2)}}\left\{1 - \frac{\exp(py) + \exp(-py)}{\exp(py_0) + \exp(-py_0)}\right\} = \\ &= \frac{C}{12k'}\left(\frac{y_0^4 - y^4}{s}\right) + \frac{a\sqrt{1+(n/2)}}{s^{2+(n/2)}} \cdot \\ &\quad \cdot \left[1 - \frac{\exp(py)+\exp(-py)}{\exp(py_0)}\left\{\sum_{m=0}^{\infty}(-1)^m \exp(-2pmy_0)\right\}\right] \\ &= \frac{C}{12k'}\left(\frac{y_0^4 - y^4}{s}\right) + \frac{a\sqrt{1+(n/2)}}{s^{2+(n/2)}} \cdot \\ &\quad \cdot \left\{1 - \sum_{m=0}(-1)^m\left[\exp(-p(\overline{2m+1}\,y_0+y)) + \exp(-p(\overline{2m+1}\,y_0 - y))\right]\right\}.\end{aligned}\right.$$

Now applying LAPLACE inversion theorem, we have

(9)
$$\left\{\begin{aligned}T = \frac{C}{12k'}(y_0^4 - y^4) + \frac{a\, t^{1+(n/2)}}{1+n/2}\Bigg\{ &-1(\sqrt{2+(n/2)})\, 2^{n+2}\sum_{m=0}^{\alpha}(-1)^m \cdot \\ &\cdot\left[i^{n+2}\operatorname{erfc}\left(\frac{\overline{2m+1}\,y_0 + y}{2\sqrt{k' t}}\right) + i^{n+2}\operatorname{erfc}\left(\frac{\overline{2m+1}\,y_0 - y}{2\sqrt{k' t}}\right)\right]\Bigg\}\end{aligned}\right.$$

## Part II.

**4** – Let us now assume

$$\frac{1}{\varrho C_v}\frac{\partial Q}{\partial t} = a \sin b\, t\,, \tag{10}$$

then equation (1) becomes,

$$\frac{\partial T}{\partial t} = a \sin bt + k'\frac{\partial^2 T}{\partial y^2} + Cy^2\,. \tag{11}$$

Let $\bar{T} = \int_0^\infty \exp((-st)T)\,dt$, be the LAPLACE transform of $T$ and let $T_0$ be the initial value of $T$. Multiplying (11) by $\exp(-st)$ and integrating between the limits 0 and $\infty$, we get

$$\frac{\partial^2 \overline{T}}{\partial y^2} - p^2 \overline{T} = -\frac{1}{k'}\left[ T_0 + \frac{Cy^2}{s} + \frac{ab}{s^2+b^2} \right], \tag{12}$$

where $p^2 = s/k'$.

Here $T_0 = (C/12k)(y_0^4 - y^4)$ as obtained in § 3. The solution of equation (12) under the boundary conditions (7) is

$$\overline{T} = \frac{C}{12k'}\left(\frac{y_0^4 - y^4}{s}\right) + \frac{ab}{s(s^2+b^2)}\left\{ 1 - \frac{\operatorname{Cosh} py}{\operatorname{Cosh} py_0} \right\}. \tag{13}$$

Now applying LAPLACE inversion theorem, we get,

$$(14)\quad \left\{ \begin{aligned} T &= \frac{C}{12k'}(y_0^4 - y^4) - \frac{a}{b}\cos bt + \\ &+ \frac{a}{b}\left[ \frac{\left\{ f_1(y)f_1(y_0) + f_2(y)f_2(y_0) \right\}\cos bt + \left\{ f_1(y)f_2(y_0) - f_2(y)f_1(y_0) \right\}\sin bt}{f_1^2(y_0) + f_2^2(y_0)} \right] \\ &- \frac{2\pi^3 a}{b}\sum_{n=0} \frac{(n+\frac{1}{2})^3(-1)^n \operatorname{Cos}(n+\frac{1}{2})\pi(y/y_0)}{(n+\frac{1}{2})^4\pi^4 + (b^2 y_0^4)/k'^2} \exp\left\{ \left(n+\frac{1}{2}\right)^2 \pi^2 \frac{k' t}{y_0^2} \right\} \\ &+ \frac{2a}{b\pi}\sum_{n=0} \frac{(-1)^n \operatorname{Cos}(n+\frac{1}{2})(\pi y/y_0)}{(n+\frac{1}{2})} \exp\left\{ \left(n+\frac{1}{2}\right)^2 \pi^2 \frac{k' t}{y_0^2} \right\}, \end{aligned} \right.$$

where

$$f_1(y) = \cosh y\sqrt{\frac{b}{2k'}}\cos y\sqrt{\frac{b}{2k'}}, \quad f_2(y) = \sinh y\sqrt{\frac{b}{2k'}}\sin y\sqrt{\frac{b}{2k'}}.$$

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## Abstract

*In this paper expressions for the temperature distributions in a channel bounded by two parallele flat plates in Poiseuille flow are derived when viscous incompressible fluid is flowing through it. Effects of viscous dissipation are not neglected and the rate of heat generation per unit volume* (i) *varies as* $(n/2)$*-th power of time, and* (ii) *is a sinusoidal function of time.*

*It is important to note that in the case of gases we have to take into account the work of compression along with viscous heating, which has not been discussed here.*

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