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Abel Continuity of Generalized Lototsky Means. (**)

A natural way to generalize the classical continuity theorem of ABEL is to consider summability methods which satisfy the following

Definition. A summability method A is said to be ABEL continuous if given a series $\sum_{n=0}^{\infty} s_n$ which is A-summable to a sum s (denoted $A(\sum_{n=0}^{\infty} s_n) = s$) then for $0 < t < 1$ the series $\sum_{n=0}^{\infty} s_n t^n$ is A-summable and $\lim_{t\to 1} A(\sum_{n=0}^{\infty} s_n t^n) = s$.

We show in this Note that a certain class of regular summability methods is ABEL continuous. Specifically, let $\{p_n\}$ be a sequence of numbers with $0 < p_{n+1} \leqslant p_n \leqslant 1$ for all n and $\sum_{n=1}^{\infty} p_n = \infty$. A sequence $\{a_n\}$ is said to be $E(p_n)$-summable to a if $\lim a'_n = a$ where

$$a'_0 = a_0; \qquad a'_n = \prod_{i=1}^{n} ((1-p_i) + p_i E)\, a_0$$

and E is the shift operator given by $Ea_n = a_{n+1}$. A series is said to be $E(p_n)$-summable if its sequence of partial sums is $E(p_n)$-summable. By setting $d_n = (1-p_n)/p_n$, one sees that the methods $E(p_n)$ are specializations of the

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methods $[F, d_n]$ of JAKIMOVSKI. For basic properties of the methods the reader is referred to [**1**] and [**2**].

If $p_n = p$ for all n, then $E(p_n)$ is the EULER-KNOPP transformation. KNOPP [**4**] proved that these methods are ABEL continuous. For $p_n = k/(n+k)$, where k is a fixed positive integer, one has the methods of STERLING-KARAMATA-LOTOTSKY [**3**], [**5**]. Of course, if $p_n = 1$ for all n, then the ABEL continuity of $E(p_n)$ reduces to the classical ABEL theorem.

The proof of the continuity theorem will follow closely the proof of the elementary ABEL theorem, but first we need a preliminary result.

Lemma. *If the $E(p_n)$-transform of $\{s_n\}$ satisfies $|s_n'| \leqslant M$, then the transform of $\{s_n t^n\}$ satisfies $|(s_n t^n)'| \leqslant M(t^n)'$.*

Proof. Let $[a_{nk}]$ and $[b_{nk}]$ be the lower triangular matrices representing $E(p_n)$ and $E(p_n)^{-1}$ respectively, then

$$a_{n+1,k+1} = (1 - p_{n+1})\, a_{n,k+1} + p_{n+1}\, a_{nk}\,, \qquad (a_{n,-1} = 0) \tag{1}$$

and

$$b_{n+1,k+1} = (1 - p_{k+2}^{-1})\, b_{n,k+1} + p_{k+1}^{-1}\, b_{nk}\,, \qquad (\text{see } [\mathbf{1}])\,. \tag{2}$$

The $E(p_n)$-transform of $\{s_n t^n\}$ is given by

$$(s_n t^n)' = \sum_{k=0}^{n} a_{nk}\, s_k\, t^k = \sum_{k=0}^{n} \sum_{i=0}^{k} a_{nk}\, b_{ki}\, s_i'\, t^k = \sum_{i=0}^{n} s_i'\, A_{ni}(t)\,, \text{ where } A_{ni}(t) = \sum_{k=i}^{n} a_{nk}\, b_{ki}\, t^k.$$

A routine calculation using (1) and (2) gives

$$A_{n+1,\,i+1}(t) = [(1 - p_{n+1}) + p_{n+1}(1 - p_{i+2}^{-1})\, t]\, A_{n,i+1}(t) + p_{n+1} p_{i+1}^{-1} A_{ni}(t)\,.$$

Since $\{p_n\}$ is non-increasing and $0 < t < 1$, it follows by induction that $A_{nk}(t) \geqslant 0$ for all n and k. An inductive argument using (2) (note that $b_{00} = 1$) shows that $\sum_{i=0}^{k} b_{ki} = 1$ for all k, thus we have

$$|(s_n t^n)'| \leqslant M \sum_{i=0}^{n} \sum_{k=i}^{n} a_{nk}\, b_{ki}\, t^k = M \sum_{k=0}^{n} t^k \sum_{i=0}^{k} a_{nk}\, b_{ki} = M \sum_{k=0}^{n} a_{nk}\, t^k = M(t^n)'.$$

Remark 1. By the regularity of $E(p_n)$, it follows that if $\{s_n\}$ is $E(p_n)$-summable, then $\{s_n t^n\}$ is $E(p_n)$-summable to zero for $0 < t < 1$.

Remark 2. In [1] it is shown that if $\sum_{n=0}^{\infty} a_n$ is $E(p_n)$-summable, then $E(p_n)(\sum_{n=0}^{\infty} a_n) = \sum_{n=0}^{\infty} p_{n+1} a_n'$ and $\lim a_n' = 0$. Conversely, if $\sum_{n=0}^{\infty} p_{n+1} a_n'$ converges and $\lim a_n' = 0$, then $\sum_{n=0}^{\infty} a_n$ is $E(p_n)$-summable with sum $\sum_{n=0}^{\infty} p_{n+1} a_n'$ (see [1], the proof of (4.1)). Thus if $\sum_{n=0}^{\infty} s_n$ is $E(p_n)$-summable, then for suitable M,

$$\sum_{n=0}^{\infty} p_{n+1} |(s_n t^n)'| \leqslant M \sum_{n=0}^{\infty} p_{n+1} (t^n)' = M E(p_n)(\sum_{n=0}^{\infty} t^n) = M/(1-t)$$

and hence $\sum_{n=0}^{\infty} p_{n+1}(s_n t^n)'$ converges. By the previous remark $\lim (s_n t^n)' = 0$ and therefore $\sum_{n=0}^{\infty} s_n t^n$ is $E(p_n)$-summable.

Theorem. *If* $0 < p_{n+1} \leqslant p_n \leqslant 1$ *for all* n *and* $\sum_{n=1}^{\infty} p_n = \infty$, *then the method* $E(p_n)$ *is Abel continuous.*

Proof. Suppose $E(p_n)(\sum_{n=0}^{\infty} s_n) = s$ and let $u_{-1} = 0$,

$$u_n = \sum_{k=0}^{n} s_k \qquad \text{and} \qquad f(t) = E(p_n)(\sum_{n=0}^{\infty} s_n t^n).$$

Since

$$\sum_{n=0}^{m} s_n t^n = (1-t) \sum_{n=0}^{m-1} u_n t^n + u_m t^m$$

and since $\{u_m t^m\}$ is $E(p_n)$-summable to 0 by Remark 1, it follows that

$$s - f(t) = (1-t) E(p_n)(\sum_{n=0}^{\infty} (s - u_n) t^n).$$

Given $\varepsilon > 0$, choose N so that $|(s-u_n)'| < \varepsilon$ for $n > N$, then by the lemma and the representation given in Remark 2 we have

$$|E(p_n)(\sum_{n=0}^{\infty} (s-u_n) t^n)| < |\sum_{n=0}^{N} p_{n+1}[(s-u_n) t^n]'| + \varepsilon \sum_{n=0}^{\infty} p_{N+n+1} (t^{N+n+1})' =$$

$$= |\sum_{n=0}^{N} p_{n+1}[(s-u_n) t^n]'| + \varepsilon t^{N+1} E(p_{N+n})(\sum_{n=0}^{\infty} t^n).$$

Since $E(p_{N+n})$ is regular we have $E(p_{N+n})(\sum_{n=0}^{\infty} t^n) = 1/(1-t)$ and hence

$$|s - f(t)| \leqslant (1-t) \,|\sum_{n=0}^{N} p_{n+1}[(s-u_n)t^n]'\,| + \varepsilon t^{N+1}$$

from which the assertion follows.

It would be interesting to learn if there are general conditions on a summability method which will guarantee that it is ABEL continuous.

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Sommaire

Nous donnons une preuve de la généralisation du théorème de la continuité d'Abel qui resulte de la substitution des sommes généraux d'un type introduit par M. Jakimovski pour les sommes dans le théorème classique.

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