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On Series of Generalised Hypergeometric Functions. (**)

Introduction.

In this paper we have established a transformation of an infinite series for FOX's H-function. It has been shown such transformations can be employed to obtain series of H-function, MEIJER's G-function [1] and MACROBERT's E-function [5].

The H-function introduced by FOX ([4], p. 408), will be represented and defined as follows:

$$(1.1)\quad \begin{cases} H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} (a_1, e_1), \ldots, (a_p, e_p) \\ (b_1, f_1), \ldots, (b_q, f_q) \end{matrix}\right] = \\ \qquad = \dfrac{1}{2\pi i} \displaystyle\int_L \dfrac{\prod_{j=1}^{m} \Gamma(b_j - f_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + a_j s)_z{}^s}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + f_j s) \prod_{j=m+1}^{p} \Gamma(a_j - e_j s)}\, \mathrm{d}s, \end{cases}$$

where an empty product is interpreted as 1, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; $e's$ and $f's$ are all positive; L is a suitable contour of BARNES type such that the poles of $\Gamma(b_j - f_j s)$, $j = 1, \ldots, m$ lie on the right hand side of the contour and those of $\Gamma(1 - a_j + e_j s)$, $j = 1, \ldots, n$ lie on the left hand side of the contour.

In what follows for sake of brevity (a_p, e_p) denotes $(a_1, e_1), \ldots, (a_p, e_p)$.

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2. - The transformation.

The transformation is:

$$(2.1)\quad \begin{cases} \sum_{r=0}^{\infty} \frac{x^r}{r!\,\Gamma(\alpha+r)} H_{p+2,q}^{m,n}\left[x \left| \begin{matrix} (a_p, e_p), (\varrho-r, h), (\sigma-r, h) \\ (b_q, f_q) \end{matrix} \right.\right] = \\ = x^{\varrho-1} \sum_{r=0}^{\infty} \frac{1}{r!}\left(1-\frac{1}{x}\right)^r H_{p+7,\ q+4}^{m+1,\ n+3}\left[zx^{-h} \,|\, I_1\right] + \\ + x^{1-\alpha-\varrho}(1-x)^{\alpha+\varrho+\sigma-2} \sum_{r=0}^{\infty} \frac{1}{r!}\left(1-\frac{1}{x}\right)^r H_{p+6,\ q+5}^{m+3,\ n+1}\left[xx^h(1-x)^{-2h} \,|\, I_2\right], \end{cases}$$

where h is an positive number $|\arg x| < \pi$

$$I_1 = \left[\begin{matrix} (\varrho-r, h),\ (\varrho+\alpha-1-r, h),\ (\alpha+\varrho+\sigma-2, 2h),\ (a_p, e_p),\ (\varrho, h),\ (\sigma, h), \\ (\alpha+\varrho-1, h),\ (\alpha+\sigma-1, h) \\ (\alpha+\varrho+\sigma-2, 2h),\ (b_q, f_q),\ (\varrho, h),\ (\varrho+\alpha-1, h),\ (\alpha+\varrho+\sigma-2-r, 2-h) \end{matrix}\right]$$

and

$$I_2 = \left[\begin{matrix} (\varrho+\sigma+\alpha-1,\ 2h),\ (a_p, e_p),\ (\varrho, h),\ (\sigma, h),\ (\alpha+\varrho-1, h),\ (\varrho, h), \\ (\alpha+\varrho+\sigma-1+r,\ 2h) \\ (\alpha+\varrho+r-1, 2h),\ (\varrho+r, h),\ (\alpha+\varrho+\sigma-1, 2h),\ (b_q, f_q),\ (\varrho, h),\ (\sigma, h) \end{matrix}\right].$$

Proof. Expressing the H-function in the left hand side of (2.1) as a MELLIN-BARNES type integral (1.1), using ([7], p. 32, (9)) we have

$$\sum_{r=0}^{\infty} \frac{x^r}{r!\,\Gamma(\alpha+r)} \cdot \frac{1}{2\pi i}\int_L A \frac{(1-\varrho+hs)_r(1-\sigma+hs)_r}{\Gamma(\varrho-hs)\,\Gamma(\sigma-hs)} z^s ds,$$

where

$$A = \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s) \prod_{j=1}^{n} \Gamma(1-a_j+a_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+f_j s) \prod_{j=n+1}^{q} \Gamma(a_j - e_j s)}.$$

Interchanging the order of integration and summation ([2], p. 500), we get

$$\frac{1}{\Gamma(\alpha)\, 2\pi i} \int_L A \frac{z^s}{\Gamma(\varrho - hs)\,\Gamma(\sigma - hs)}\, {}_2F_1\left[\begin{matrix}1-\varrho+hs,\ 1-\sigma+hs;\ x\\ \alpha\end{matrix}\right] ds\,.$$

Now from ([3], p. 109, (4)), the expression becomes

$$\frac{1}{2\pi i}\int_L A \frac{\Gamma(\alpha+\varrho+\sigma-2-2hs)\,x^{-1+\varrho-hs}}{\Gamma(\varrho-hs)\,\Gamma(\sigma-hs)\,\Gamma(\alpha-1+\varrho-hs)\,\Gamma(\alpha-1+\sigma-hs)}\ \cdot$$

$$\cdot\ {}_2F_1\left[\begin{matrix}1-\varrho+hs,\ 2-\varrho-\alpha+hs;\ 1-(1/x)\\ 3-\varrho-\sigma-\alpha+2hs\end{matrix}\right] z^s\,ds\ +$$

$$+\frac{1}{2\pi i}\int_L A \frac{\Gamma(2-\varrho-\sigma-\alpha+2hs)\,x^{1-\varrho-\alpha+hs}\,(1-x)^{\alpha+\varrho+\sigma-2-2hs}}{\Gamma(\varrho-hs)\,\Gamma(\sigma-hs)\,\Gamma(1-\varrho+hs)\,\Gamma(1-\sigma+hs)}\ \cdot$$

$$\cdot\ {}_2F_1\left[\begin{matrix}\alpha+\varrho-1-hs,\ \varrho-hs;\ 1-(1/x)\\ \alpha+\varrho+\sigma-1-2hs\end{matrix}\right] z^s\,ds\,.$$

Expressing the hypergeometric functions as infinite series and interchanging the order of summation and integration, we obtain

$$\sum_{r=0}^{\infty} \frac{x^{\varrho-1}(1-(1/x))^r}{r!}$$

$$\int_L A \frac{\Gamma(\alpha+\varrho+\sigma-2-2hs)\,\Gamma(1-\varrho+r+hs)\,\Gamma(2-\alpha-\varrho+r+hs)\,\Gamma(3-\alpha-\varrho-\sigma+2hs)}{\Gamma(\varrho-hs)\,\Gamma(\sigma-hs)\,\Gamma(\alpha+\varrho-1-hs)\,\Gamma(\alpha+\sigma-1-hs)\,\Gamma(1-\varrho+hs)\,\Gamma(2-\varrho-\alpha+hs)}$$

$$\times \frac{x^{-hs} z^s}{\Gamma(3-\alpha-\varrho-\sigma+r+2hs)}\, ds\ +$$

$$+\sum_{r=0}^{\infty} \frac{x^{1-\alpha-\varrho}(1-x)^{\alpha+\varrho+\sigma-2}}{r!}\left(1-\frac{1}{x}\right)^r\ \cdot$$

$$\cdot\ \frac{1}{2\pi i}\int_L A \frac{\Gamma(2-\varrho-\sigma-\alpha+2hs)\,\Gamma(\alpha+\varrho-1+r-hs)\,\Gamma(\varrho+r-hs)}{\Gamma(\varrho-hs)\,\Gamma(\sigma-hs)\,\Gamma(1-\varrho+hs)\,\Gamma(1-\sigma+hs)\,\Gamma(\alpha+\varrho-1-hs)\,\Gamma(\varrho-hs)}$$

$$\times \frac{\Gamma(\alpha+\varrho+\sigma-1-2hs)\,x^{hs}\,(1-x)^{-2hs} z^s}{\Gamma(\alpha+\varrho+\sigma-1+r-2hs)}\, ds\,.$$

On applying (1.1), the right hand side of (2.1) is obtained.

3. - Particular cases.

In (2.1), putting $x=1$ it reduces to the form

$$(3.1)\quad \begin{cases} \sum_{r=0}^{\infty} \frac{1}{r!\,\Gamma(\alpha+r)} H_{p+2,q}^{m,n}\left[x\left|\begin{matrix}(a_p, e_p),\ (\varrho-r, h),\ (\sigma-r, h)\\ (b_q, f_q)\end{matrix}\right.\right] = \\ = H_{p+4,\ q+1}^{m+1,\ n} \cdot \left[x\left|\begin{matrix}(a_p, e_p),\ (\varrho, h),\ (\sigma, h),\ (\alpha+\varrho-1, h),\ (\alpha+\sigma-1, h)\\ (\alpha+\varrho+\sigma-2, 2h),\ (b_q, f_q)\end{matrix}\right.\right], \end{cases}$$

where h is positive number.

In (3.1), assuming h as a positive integer, putting $e_j = f_i = 1$ ($j = 1, ..., p$; $i = 1, ..., q$), using the formula

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_p, 1)\\ (b_q, 1)\end{matrix}\right.\right] = G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right]$$

and simplifying with the help of (1.1), ([3], p. 4, (11)) and ([3], p. 207, (1)), we obtain ([1], (3.1)), viz.

$$\sum_{r=0}^{\infty} \frac{h^{2r}}{r!\,\Gamma(\alpha+r)} G_{p+2h,q}^{m,n}\left[z\left|\begin{matrix}a_p,\ \Delta(h, \varrho-r),\ \Delta(h, \sigma-r)\\ b_q\end{matrix}\right.\right] =$$

$$= \frac{2^{\alpha+\varrho+\sigma-3}}{\sqrt{(\pi)}\, h^{\alpha-1/2}} G_{p+4h,\ q+2h}^{m+2h,\ n} \cdot$$

$$\cdot \left[z\, 2^{-2h}\left|\begin{matrix}a_p,\ \Delta(h, \varrho),\ \Delta(h, \sigma),\ \Delta(h, \alpha+\varrho-1),\ \Delta(h, \alpha+\sigma-1)\\ \Delta(2h, \alpha+\varrho+\sigma-2),\ b_q\end{matrix}\right.\right],$$

where the symbol $\Delta(h, a)$ represents the set of parameters

$$\frac{a}{h},\ \frac{a+1}{h},\ \dots,\ \frac{a+h-1}{h}.$$

In (3.1), setting $m = q = p$, $n = 1$, $p = q + 1$, $e_j = f_i = 1$ ($j = 1, ..., p$; $i = 1, ..., q$), $a_1 = 1$, replacing a_{j+1} by a_j ($j = 1, ..., q$) using the formula

$$H_{q+1,p}^{p,1}\left[z\left|\begin{matrix}(1, 1)\,(b_q, 1)\\ (a_p, 1)\end{matrix}\right.\right] = E\left[\begin{matrix}a_p : z\\ b_q\end{matrix}\right],$$

and simplifying with the help of (1.1), ([**3**], p. 4, (11)) and ([**6**], p. 374, (36)), we get a known result obtained by MACROBERT ([**5**], p. 135, (15)).

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