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Borel Measures on Connected Door Spaces. (**)

Introducton.

Let a linear space F of complex valued functions f defined on a set X be given. For x_0 fixed in X, the DIRAC delta functional at x_0, δx_0, mapping F into the set of complex numbers, is given by

$$\delta x_0(f) = \int f \, dm \qquad \text{for each } f \text{ in } F,$$

where m is the point measure at x_0 defined on the measurable space $(X, 2^X)$. One observes that relative to the distinguished point topology on X at x_0, m is a BOREL measure which in particular is constant on the proper open subsets of X. The distinguished point topology is frequently the source of useful counter examples in point set topology. It is also a special case of a connected « DOOR » topology. A DOOR topological space is one in which each subset of the space is either open or closed. Connected DOOR spaces have been utilized in the representation of logical connectives as mappings between appropriate topological spaces.

In this paper we consider the question, « Does every connected DOOR space admit a non-zero BOREL measure which is constant on proper open sets? ». As a consequence of a classification theorem that we establish for connected DOOR spaces, an affirmative answer is obtained.

By employing an interesting construction afforded by ZORN's lemma, and utilizing the known result, that if X is a set with cardinality C, with C infinite,

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then there is a subset A of 2^X with cardinality 2^C such that for any finite subset $S_1, \ldots, S_n$ of A and any sequence $e_1, e_2, \ldots$ of 1 s and -1's,

$$\bigcap_{k=1}^{n} S_{k,e_k} \neq \emptyset \quad \text{where} \quad S_{k,1} = S_k \quad \text{and} \quad S_{k,-1} = X - S_k ,$$

we determine the cardinality of the collection of all connected Door topologies on a set X with a given cardinality. In particular we show that the set of all connected Door topologies on X has the same cardinality as the set of all topologies, on X if X is infinite, namely, 2^{2^C}, where C is the cardinality of X.

Theorem 1. (Classification Theorem). *If (X, T) is a connected Door space and $X \neq \emptyset$, then one of the following is true:*

(1) *There is a p in X such that A is a member of T if and only if either p is not a member of A or $A = X$.*

(2) *There is a p in X such that A is a member of T if and only if p is a member of A or $A = \emptyset$.*

(3) *(X, T) is a T_1 space such that if A_1 and A_2 are open and $A_1 \neq \emptyset \neq A_2$, then $A_1 \cap A_2$ is an infinite subset of X.*

Note that the condition (3) implies that the space is not T_2. In fact, Theorem 1 shows that there are no T_2 connected Door spaces except for the 0 and 1 member spaces. The cases in Theorem 1 are mutually exclusive except that (1) and (2) are both true for 1 and 2 member spaces.

Corollary. *Given an arbitrary connected Door space X, then there exists on X a non-zero Borel measure which is constant on the collection of proper open subsets of X, those open sets which are neither empty nor the whole space.*

Theorem 2. (Cardinality Theorem). *If X is a set with cardinality $C > 0$, then there are C connected Door topologies of type (1) and C connected Door topologies of type (2) for X. If C is finite, there are no topologies of type (3). If C infinite, there are 2^{2^C} connected Door topologies of type (3) for X.*

Consequently, if X is infinite, the set of all connected Door topologies on X has the same cardinality as the set of all topologies on X.

To prove Theorem 1 the following lemma will be needed.

Lemma. *If (X, T) is a connected Door space and G and H are disjoint, non empty, open subsets of X, then X contains exactly one closed singleton.*

There exists at least one closed singleton, p; otherwise the topology would be discrete and since there are at least two points in X, the space would not be connected.

If there were a second closed singleton, q, let $G_1 = G - \{p, q\}$ and $H_1 = H - \{p, q\}$. Then G_1 and H_1 are open and non empty (since otherwise G or H would be closed) and $G_1 \cap H_1 = \emptyset$. Since $G_1 = (G_1 \cup \{p\}) \cap (G_1 \cup \{q\})$ and is not closed, either $G_1 \cup \{p\}$ or $G_1 \cup \{q\}$ is open. Without loss of generality assume that $G_1 \cup \{p\}$ is open. Since $\{p\} = (G_1 \cup \{p\}) \cap (H_1 \cup \{p\})$ is not open, $H_1 \cup \{p\}$ must be closed. Similarly $H_1 \cup \{q\}$ is open and $G_1 \cup \{q\}$ is closed. Then $(G_1 \cup \{q\}) \cup (H_1 \cup \{p\}) = (H_1 \cup \{q\}) \cup (G_1 \cup \{p\})$ is both open and closed and therefore must equal X. Consequently, $G_1 \cup \{p\}$ and $H_1 \cup \{q\}$ disconnect X; this contradiction shows that q can not exist.

Theorem 1 will now be proved by considering 3 cases: (1), (X, T) contains a pair of disjoint, non empty, open subsets; (2), (X, T) contains exactly one open singleton and does not contain a pair of disjoint, non empty, open subsets; (3) (X, T) contains no open singletons and does not contain a pair of disjoint, non empty, open subsets. The statements in Theorem 1 will be shown to be true for the correspondingly numbered cases.

In case (1), the lemma implies that there is exactly one closed singleton, p. If $p \notin G$, then G is a union of open singletons and G is open. If G is open and not X or $\emptyset$, then $\tilde{G} = X - G$ is closed and not open. Therefore $\tilde{G}$ is not the union of open singletons so that $p \in \tilde{G}$. Therefore, the space satisfies statement (1) of Theorem 1.

In case (2) let p be the open singleton and let G be a non empty open set. Since $\{p\}$ and G can not be a pair of non empty disjoint open sets, $p \in G$. Conversely, if $p \in G$, then $p \notin \tilde{G}$. Therefore, $\tilde{G}$ is closed or $\emptyset$. Consequently G is open. Consequently, statement (2) is valid.

In case (3) the space is certainly T_1 since $\{\tilde{p}\}$ is open for every point, p, in X. The space is infinite; otherwise since every singleton is closed the space would be discrete. If the intersection of two open sets were finite but not empty, this intersection would be both open and closed and disconnect the space. The intersection of two open sets can not be empty unless one of the sets is empty by the definition of this case. Therefore, statement (3) is true.

This completes the proof of Theorem 1.

The only part of Theorem 2 that is not immediate is the statement that if C is infinite, then there are 2^{2^C} connected DOOR topologies of type (3). To prove this, let X be a set with cardinality C, with C infinite. By [**1**], p. 45 there is a subset, A of 2^X with cardinality 2^C such that for any finite subset, $\{S_1, \ldots, S_n\}$ of A and any sequence, $e_1, e_2, \ldots$ of 1's and -1's,

$$\bigcap_{k=1}^{n} S_{k,e_k} \neq \emptyset \qquad \text{where} \qquad S_{k,1} = S_k \quad \text{and} \quad S_{k,-1} = \tilde{S}_k .$$

Now for each p in X, choose a set R_p such that $p \notin R_p$ and either $R_p \in A$ or $\tilde{R}_p \in A$, and for $q \in X$ with $q \neq p \cdot R_q \neq R_p$. Now let B consist of all the R_p's and for each set in A which is neither a R_p or a $\tilde{R}_p$ choose arbitrarily either the set or its complement but not both to be a member of B. Since this arbitrary choice is made for 2^c sets, there are 2^{2^c} possible choices of B.

Given a collection, B, let D be the set of all S such that there exists a finite collection, $T_1, \ldots, T_n$ of members of B such $S \supset \bigcap_{k=1}^{n} T_k$. Then consider all collections, $G \subset 2^X$, such that

1. $D \subset G$.
2. $S \in G$ and $T \in G$ implies that $S \cap T$ is infinite.
3. $S_1, \ldots, S_n$ in G and $\bigcap_{k=1}^{n} S_k \subset T$ imply that $T \in G$.

First it will be shown that D satisfies these conditions. The first and third are immediate. If there were S and T in D such that $S \cap T = \{p_1, \ldots, p_n\}$ were finite, let $S_1, \ldots, S_m$ and $T_1, \ldots, T_k$ be members of B such that $S \supset \bigcap_{i=1}^{m} S_i$ and $T \supset \bigcap_{i=1}^{k} T_j$. Then $\bigcap_{j=1}^{m} S_i \cap \bigcap_{j=1}^{k} T_j \cap \bigcap_{i=1}^{n} R_{p_i} = \emptyset$. This contradicts the definition of A.

By ZORN's lemma there exists a maximal chain of such $G's$ linearly ordered by inclusion. Let G^* be the union of such a chain. It is immediate that G^* satisfies the conditions 1-3. It will now be shown that $G^* \cup \{\emptyset\}$ is a connected DOOR topology for X.

$G^* \cup \{\emptyset\}$ is a topology on X by property 3. It is a connected topology by property 2.

If $S \notin G^*$, let G_1 be the set of all H such that there exists T in G^* with $S \cap T \subset H$. Then G_1 satisfies conditions 1 and 3 and properly contains the maximal set G^*. Therefore, G_1 can not satisfy condition 2 and there exist T_1 and T_2 in G^* such that $T_1 \cap T_2 \cap S = \{p_1, \ldots, p_n\}$ is finite. Then $S \cap T_1 \cap \cap T_2 \cap \bigcap_{i=1}^{n} R_{p_i} = \emptyset$ so that $T_1 \cap T_2 \cap \bigcap_{i=1}^{n} R_{p_i} \subset \tilde{S}$. Therefore $\tilde{S} \in G^*$ and $G^* \cup \{\emptyset\}$ is a DOOR topology.

Since each different B will produce a different G^*, this proves that there are 2^{2^c} connected DOOR topologies on X. Since there are only C of types (1) and (2), there are 2^{2^c} of type (3).

Proof of Corollary of Theorem 1. Let a connected DOOR space (X, T), $X \neq 0$, be given. Then by Theorem 1, (X, T) satisfies condition (1), (2) or (3). If (X, T) satisfies condition (1), there is a p in X such that A is open if and only if p is not a member of A or $A = X$. In this case the required BOREL measure is the point measure at p which indeed vanishes on proper open sets.

If (X, T) satisfies condition (2), there is a p in X such A is open if and only if p is a member of A or $A = \emptyset$. In this case the point measure at p provides a BOREL measure whose value on each proper open set is 1.

If (X, T) satisfies condition (3), then (X, T) is a T_1 space such that if A_1 and A_2 are open and $A_1 \neq \emptyset \neq A_2$, then $A_1 \cap A_2$ is an infinite subset of X. In this case a measure of the required type is given by the « counting » measure m mapping 2^X into the extended real numbers and defined by $m(A) =$ the cardinality of A if A is finite, and $m(A) = \infty$ if A is infinite, for each subset A of X. It is clear that m is a measure on X which is constant on the collection of proper open sets, assuming the value ∞ on each. In the sequel we show that each compact subset of X is finite. It follows that m is finite on compact sets and therefore is a BOREL measure.

Lemma. *Let (X, T) satisfy condition (3) of Theorem 1. Suppose the subset A of X is compact. Then A is finite.*

Proof of Lemma. Suppose A is infinite. Then clearly there exist subsets G, H, of X satisfying:

(i) $$G \cup H = A$$

(ii) $$G \cap H = \emptyset$$

(iii) Each of G, H, is infinite.

One of the subsets G, H, is closed. Certainly each is open or closed since (X, T) is a DOOR space. However, if both are open, the (ii) above contradicts condition (3) of Theorem 1. We assume without loss of generality that H is closed. Let

$$D = \{x_1, x_2, \ldots\}$$

be a denumerable subset of H. D is closed, for otherwise $D \cap X - H$ is open and satisfies $D \cap X - H = \emptyset$ which contradicts condition (3). Let $E = H - D$, then $X - H \cup E = X - D$ is open. We conclude in like manner that for $D_n = D - \{x_1, x_2, \ldots, x_n\}$ and $E_n = H - D_n$, that $X - H \cup E_n$ is open for each positive integer n. It is clear now that the collection of all sets of the form $X - H \cup E_n$ is an open cover of A which does not have a finite subcover. Hence A is not compact.

References.

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Abstract.

In this paper we give a complete classification of « Connected Door Spaces ». Subsequently we utilize these classification results to establish the existence of non-trivial Borel measures on Connected Door Spaces with the distinctive property of being constant on the family of proper open sets, that is the collection of open sets which are neither the empty set nor the whole space.

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