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Unique Factorization in a 2-fir With Right ACC_1. (**)

1. - Introduction.

A 2-fir (also called a weak BEZOUT ring) is an integral domain in which the sum and intersection of any two principal right ideals is principal whenever the intersection is nonzero. Then the principal right ideals containing a fixed non-zero element form a sublattice of the lattice of all right ideals. In [**1**] it was noted that any factorization of $c \in R$ corresponds to a chain of strictly cyclic submodules of R/cR (i.e. modules with one generator and one irredundant defining relation). This suggests that we operate in the category $C = C_R$ of all strictly cyclic right R-modules and all homomorphisms. Any module in this category has the form R/aR $(a \neq 0)$. A 2-fir with right ACC_1 is one which satisfies the ascending chain condition for principal right ideals. For a discussion of unique factorization of the nonzero elements of a principal right ideal domain see [**2**].

2. - Right denominator set.

Suppose R is any ring and S is a subsemigroup of R (qua multiplicative semigroup). We call S a right denominator set if S satisfies the following conditions:

(1) $s, t \in S$ implies $st \in S$.
(2) $1 \in S$.
(3) Given $a \in R$, $s \in S$ there exist $a_1 \in R$, $s_1 \in S$ such that $as_1 = sa_1$.
(4) If $ua = 0$ for some $u \in S$ and $a \in R$, then there exists $v \in S$ such that $av = 0$.

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If S is a right denominator set, we can define a ring of fractions R_S by taking equivalence classes of expressions as^{-1} ($a \in R$, $s \in S$). We can write $as^{-1} = bt^{-1}$ whenever there exist $u, v \in S$ with $au = bv$ and $su = tv$. The product $as^{-1}bt^{-1}$ is defined by first finding b_1, t_1 such that $sb_1 = bs_1$ (so that $s^{-1}b = b_1s_1^{-1}$) and then putting $as^{-1}bt^{-1} = ab_1(ts_1)^{-1}$. Any two fractions as^{-1} and bt^{-1} can be brought to a common denominator by finding s_1, t_1 such that $st_1 = ts_1 = m$ say, and observing that $as^{-1} = at_1m^{-1}$, $bt^{-1} = bs_1m^{-1}$. Now addition is defined by the rule:

$$as^{-1} + bt^{-1} = (at_1 + sb_1)m^{-1}.$$

Let

$$\Phi: R \to R_S$$

be a map that sends $a \mapsto a \cdot 1^{-1}$.

Now $as^{-1} = bt^{-1}$ whenever there exist $u, v \in S$ with $au = bv$ and $su = tv$. Set $s = t = 1$, $b = 0$. Thus $a \cdot 1^{-1} = 0$ implies $au = 0$ and $u = v$.

Therefore

$$\ker \Phi = \{a \in R \mid au = 0 \text{ for some } u \in S\}.$$

Φ is injective if $\ker \Phi = 0$, i.e. if for all $a \in R$, $au = 0$ for some $u \in S$ which implies that $a = 0$. Hence S consists of non-zero divisors.

Let M be a right R-module. We define $x \in M$ as *S-negligible* if $xs = 0$ for some $s \in S$. Let $t_S(M)$ denote the set of all S-negligible elements of M. Then $t_S(M)$ is the kernel of canonical mapping:

$$M \to M \otimes_R R_S.$$

Lemma. *$t_S(M)$ is a submodule of M and $t_S(M/t_S(M)) = 0$.*

Proof. Let $x \in t_S(M)$. This implies that $xs = 0$ for some $s \in S$. Now for any $a \in R$, $s \in S$ there exist $a_1 \in R$ and $s_1 \in S$ such that $as_1 = sa_1$. Thus $xas_1 = xsa_1 = 0$. Therefore $xa \in t_S(M)$ for all $a \in R$. Now let $x, y \in t_S(M)$. This implies that $xs = 0$, $ys_1 = 0$ for some $s, s_1 \in S$; i.e., $xst = 0$, $ys_1t_1 = 0$ for all t, t_1. Choose $t, t_1 \in S$ so that $st = s_1t_1 = n$ say. Now $n \in S$, since s, s_1, t, t_1 all $\in S$. Also $(x + y)n = xst + ys_1t_1 = 0$. Therefore $(x + y) \in t_S(M)$. Thus $t_S(M)$ is a submodule of M.

Suppose $(x + t_S(M))s = 0$ for any $x \in M$ and some $s \in S$. Then $xs \in t_S(M)$.

Hence there exists $t \in S$ such that $x s t = 0$. Since $s, t \in S$ implies that $st \in S$, it follows that $x \in t_s(M)$, i.e. $x + t_s(M) = t_s(M)$. Thus $t_s(M/t_s(M)) = 0$. This proves the lemma.

We shall call $t_s(M)$ as S-torsion part of M. It is in fact the *unique* greatest such submodule.

3 – We shall call $z \in R$ *regular* if no factor of it is a left or right zero-divisor. Clearly any factor of a regular element is again regular. In an integral domain the regular elements are just the non-zero elements. An R-module M is said to be *strictly cyclic* or a *C-module* if it has a presentation of the form $M \cong R/zR$, where z is regular. The module R/zR turns out to reflect all the properties of factorization of z itself.

Theorem 1. *Let R be a 2-fir with right ACC_1 and S a right denominator set in R. Let C be the category of strictly cyclic right R-modules. If $M \in C$, then $t_s(M) \in C$.*

Proof. Let $M = R/zR$, where $z \in R^*$ (R^* denotes the set of nonzero elements of R). Let C_s be the collection of C-submodules R/sR of R/zR such that $s \in S$. By condition (3), given $a \in R$, $s \in S$ there exist $a_1 \in R$, $s_1 \in S$ such that $as_1 = sa_1$. Hence $as_1 \in sR$ which implies that $(a + sR)s_1 = 0$. Hence $a + sR \in t_s(M)$ whence $R/sR \subseteq t_s(M)$. Since R is a 2-fir with right ACC_1, each object of C_s has the ascending chain condition. Thus we may select a (not necessarily proper) maximal member $M_0 = R/s_0R \subseteq t_s(M)$ where $z = xs_0$ for some $s_0 \in S$ and x has no nonunit right factor in S.

If $M_0 \subset t_s(M)$, then there exists M_1 cyclic, $\subseteq t_s(M)$ but $\not\subseteq M_0$. Since M_1 is cyclic and $\subseteq M$, therefore $M_1 = (bR + zR)/zR \cong bR/(bR \cap zR)$ $(b \neq 0)$. Suppose that $bR \cap zR = 0$, then $M_1 = bR/0 \cong bR \cong R$. Thus M_1 is free on u (say). But $M_1 \subseteq t_s(M)$, therefore $us = 0 (s \in S)$. Now u is free. Therefore $s = 0$ which contradicts the fact that $0 \notin S$. Hence $bR \cap zR \neq 0$. Now since R is a 2-fir, therefore, $bR \cap zR = dR$, for some $d \in R$. Then $M_1 = bR/dR \cong R/aR$ where $d = ba$. This implies that M_1 is strictly cyclic. Thus any cyclic sub-module of $t_s(M)$ is in C. Now $t_s(M)$ is a sub-module of M by the above lemma. Thus M_1 being a strictly cyclic submodule of M is in C_s. Now $M_0 = xR/zR$ and $M_1 = yR/zR$ where $z = xs_0 = ya$ i.e. $xR, yR \supseteq zR$. Since R is a 2-fir and $xR \cap yR \supseteq zR \neq 0$, therefore $xR + yR = eR$. Thus $M_0 + M_1 = (xR + yR)/zR = eR/zR$ which is again in C_s. Also $M_0 + M_1 = eR/zR \supset xR/zR = M_0$ which contradicts the maximality of M_0. Thus $M_0 \not\subset t_s(M)$. Therefore $M_0 = t_s(M)$ which implies that $t_s(M) \in C$. This proves the theorem.

4 – Let R be any ring and let $I = \{\alpha \mid 0 \leqq \alpha \leqq \alpha_0\}$ be an initial segment of ordinals. A collection $\{S_\alpha \mid \alpha \in I\}$ of right denominator sets in R is called a *right denominator chain* in R if the following conditions hold:

(1) $S_\alpha \subset S_{\alpha+1}$ for each $\alpha \in I$, $\alpha \neq \alpha_0$.

(2) $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ if α is a limit ordinal.

Theorem 2. *Let R be a 2-fir with right ACC_1. Let $I = \{\alpha \mid 0 \leqq \alpha \leqq \alpha_0\}$ be an initial segment of ordinals and let $\{S_\alpha \mid \alpha \in I\}$ be a right denominator chain in R. Let C be the category of strictly cyclic right R-modules. Each $M \in C$ has* a unique *sequence of C-submodules*

$$M \supset M_0 \supset M_1 \supset \ldots \supset M_n = 0$$

where $M_0 = t_{S_{\alpha_0}}(M)$, $M_i = t_{S_{\alpha_i - 1}}(M_{i-1})$ $(i = 1, 2, \ldots, n)$ and α_i are nonlimit ordinals such that $\alpha_0 \geqq \alpha_1 > \ldots > \alpha_n$.

Proof. Let $M = R/zR$ where $z \in R^*$. If $M_0 = 0$ then there is nothing to prove. Otherwise by Theorem 1 $M \supset a$ unique $M_0 = t_{S_{\alpha_0}}(M) = R/s_0R$, where $z = rs_0$ for some $s_0 \in S_{\alpha_0}$ and r has no nonunit right factor in S_{α_0}. Let α_1 be the least ordinal such that $s_0 \in S_{\alpha_1}$. Clearly α_1 is not a limit ordinal and $\alpha_0 \geqq \alpha_1$. It follows by Theorem 1 that $M_0 \supset a$ unique $M_1 = t_{S_{\alpha_1 - 1}}(M_0) = R/s_1R$ where $s_0 = a_{\alpha_1} s_1$ for some element $s_1 \in S_{\alpha_1 - 1}$ and a_{α_1} has no nonunit right factor in $S_{\alpha_1 - 1}$. Clearly $a_{\alpha_1} \in S_{\alpha_1 - 1}$ because $s_0 \in S_{\alpha_1}$. If $M_1 \neq 0$ let α_2 be the least ordinal such that $s_1 \in S_{\alpha_2}$. Then $\alpha_1 > \alpha_2$ and α_2 is not a limit ordinal. Another application of Theorem 1 yields $M_1 \supset a$ unique $M_2 = t_{S_{\alpha_2 - 1}}(M_1) = R/s_2R$ where $s_1 = a_{\alpha_2} s_2$ for some element $s_2 \in S_{\alpha_2 - 1}$ and a_{α_2} has no nonunit right factor in $S_{\alpha_2 - 1}$. Clearly $a_{\alpha_2} \in S_{\alpha_2 - 1}$ because $s_1 \in S_{\alpha_2}$. If $M_2 \neq 0$ we may repeat the argument. Now this process cannot continue indefinitely since we would obtain an infinite sequence $\alpha_1 > \alpha_2 > \ldots$ contradicting the well-ordering of ordinals. Thus the process stops, say, with integer n. That is, a_{α_n} has no nonunit right factor in $S_{\alpha_n - 1}$ and $M_n = 0$. This proves the theorem.

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References.

[1] P. M. Cohn, *Noncommutative unique factorization domains*, Trans. Amer. Math. Soc. **109** (1963), 313-331.

[2] R. A. BEAUREGARD, *Infinite primes and unique factorization in a principal right ideal domain,* Trans. Amer. Math. Soc. **141** (1969), 245-254.

Abstract

In this Note we define a right denominator set and construct a ring of fractions. Then we develope some general results for a 2-fir with right ACC_1.

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