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Applications of Proximate Order. (**)

1. - Preliminary remarks.

Let $G(r)$ be a positive function defined for all $r \geqslant r_0$ and let

$$\limsup_{r\to\infty} \frac{\log G(r)}{\log r} = \varrho\,, \qquad \text{where} \qquad 0 < \varrho < \infty\,.$$

Then it is possible to determine a continuous function $\varrho(r)$ defined for $r \geqslant r_0$ with the following properties:

(i) $\lim_{r\to\infty} \varrho(r) = \varrho$;

(ii) $\varrho(r)$ is differentiable for $r \geqslant r_0$ except at isolated points, at which $\varrho'(r-0)$ and $\varrho'(r+0)$ exist;

(iii) $\lim_{r\to\infty} (r \log r)\,\varrho'(r) = 0$ where $\varrho'(r)$ can be interpreted as either $\varrho'(r-0)$ or $\varrho'(r+0)$ when these are unequal;

(iv) $G(r) \leqslant r^{\varrho(r)}$ for $r \geqslant r_0$;

(v) $G(r) = r^{\varrho(r)}$ for a sequence $\{r_n\}$ of values of r increasing to ∞ with n.

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See LEVIN ([2], p. 35).

Similarly, if

$$\liminf_{r\to\infty} \frac{\log G(r)}{\log r} = \lambda, \quad \text{where} \quad 0 < \lambda < \infty,$$

then it is possible to determine a continuous function $\lambda(r)$ defined for $r \geq r_0$, with the following properties:

(i) $\lim_{r\to\infty} \lambda(r) = \lambda$;

(ii) $\lambda(r)$ is differentiable for $r \geq r_0$ except at isolated points, at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist;

(iii) $\lim_{r\to\infty} (r \log r)\lambda'(r) = 0$ where $\lambda'(r)$ can be interpreted as either $\lambda'(r-0)$ or $\lambda'(r+0)$ when these are unequal;

(iv) $G(r) \geq r^{\lambda(r)}$ for $r \geq r_0$;

(v) $G(r) = r^{\lambda(r)}$ for a sequence $\{r_n\}$ of values of r increasing to ∞ with n.

See S. M. SHAH [3].

$\varrho(r)$ is called a proximate order relative to $G(r)$ and $\lambda(r)$ is called a lower proximate order relative to $G(r)$.

We now show that the functions $\varrho(r)$ and $\lambda(r)$, with the above properties, exist even when $\varrho = 0$ or $\lambda = 0$ respectively. Suppose $\varrho = 0$ so that $\lim_{r\to\infty} (\log G(r)/\log r) = 0$. Then $\lim_{r\to\infty} (\log (rG(r))/\log r) = 1$. Hence there exists a proximate order $\varrho_1(r)$ relative to $rG(r)$.

Define $\varrho(r) = \varrho_1(r) - 1$, then $\varrho(r) \to 0$ as $r \to \infty$ since $\varrho_1(r) \to 1$, and $r \log r \varrho'(r) = r \log r \varrho_1'(r) \to 0$ as $r \to \infty$. Also $rG(r) \leq r^{\varrho_1(r)}$ so that $G(r) \leq r^{\varrho_1(r)-1} = r^{\varrho(r)}$ for $r \geq r_0$ and $rG(r) = r^{\varrho_1(r)}$ so that $G(r) = r^{\varrho(r)}$ for a sequence $\{r_n\}$ of values of r, increasing to ∞ with n.

The existence of $\lambda(r)$, when $\lambda = 0$, follows similarly. For, when $\lambda = 0$, $\liminf_{r\to\infty} (\log G(r)/\log r) = 0$ so that $\liminf_{r\to\infty} (\log (rG(r))/\log r) = 1$ and we can define $\lambda(r) = \lambda_1(r) - 1$ where $\lambda_1(r)$ is a lower proximate order relative to $rG(r)$.

In the case $\varrho > 0$, it is known that $(kr)^{\varrho(kr)} \sim k^{\varrho} r^{\varrho(r)}$ as $r \to \infty$ uniformly in k for $0 < a \leq k \leq b < \infty$. This holds when $\varrho = 0$ also, for, in this case if $\varrho_1(r)$ is a proximate order relative to $rG(r)$ and $\varrho(r) = \varrho_1(r) - 1$, then $(kr)^{\varrho_1(kr)} \sim kr^{\varrho_1(r)}$ as $r \to \infty$ uniformly in k for $0 < a \leq k \leq b < \infty$. Hence $(kr)^{\varrho_1(kr)-1} \sim r^{\varrho_1(r)-1}$ or $(kr)^{\varrho(kr)} \sim k^{\varrho} r^{\varrho(r)}$ as $r \to \infty$ uniformly in k for $0 < a \leq k \leq$

$\leqslant b < \infty$, since $\varrho = 0$. Similarly if $\lambda(r)$ is a lower proximate order relative to $G(r)$, with $0 \leqslant \lambda < \infty$, then $(kr)^{\lambda(kr)} \sim k^{\lambda} r^{\lambda(r)}$ as $r \to \infty$ uniformly in k for $0 < a \leqslant k \leqslant b < \infty$.

Again, if $\varrho > 0$, it is known that

$$\int_{r_0}^{r} t^{\varrho(t)-k}\, \mathrm{d}t \sim \frac{r^{\varrho(r)-k+1}}{-k+\varrho+1} \qquad \text{as } r \to \infty,$$

for all $k < \varrho + 1$, and

$$\int_{r}^{\infty} t^{\varrho(t)-k}\, \mathrm{d}t \sim \frac{r^{\varrho(r)-k+1}}{k-\varrho-1} \qquad \text{as } r \to \infty,$$

for all $k > \varrho + 1$.

We now show, by elementary proofs, that, if l is a positive constant, then

$$(\beta) \qquad \int_{r_0}^{lr} t^{\varrho(t)-k}\, \mathrm{d}t \sim \frac{l^{\varrho-k+1}}{\varrho-k+1}\, r^{\varrho(r)-k+1} \qquad \text{as } r \to \infty,$$

for all $k < \varrho + 1$, and

$$(\gamma) \qquad \int_{lr}^{\infty} t^{\varrho(t)-k}\, \mathrm{d}t \sim \frac{l^{\varrho-k+1}}{k-\varrho-1}\, r^{\varrho(r)-k+1} \qquad \text{as } r \to \infty,$$

for all $k > \varrho + 1$.

Let $k < \varrho + 1$. Then $\varrho(r) - k + 1 \to \varrho - k + 1 > 0$ as $r \to \infty$. Hence $r^{\varrho(r)-k+1} \to \infty$ as $r \to \infty$. Hence, by L'Hospital's rule,

$$(1) \qquad \lim_{r\to\infty} \frac{\int_{r_0}^{lr} t^{\varrho(t)-k}\, \mathrm{d}t}{\dfrac{l^{\varrho-k+1}}{\varrho-k+1}\, r^{\varrho(r)-k+1}} = \lim_{r\to\infty} \frac{(\varrho-k+1)\, \dfrac{\mathrm{d}}{\mathrm{d}r} \displaystyle\int_{r_0}^{lr} t^{\varrho(t)-k}\, \mathrm{d}t}{l^{\varrho-k+1}\, \dfrac{\mathrm{d}}{\mathrm{d}r}\left(r^{\varrho(r)-k+1}\right)},$$

provided the latter limit exists. But

$$\frac{(\varrho-k+1)\dfrac{d}{dr}\displaystyle\int_{r_0}^{lr} t^{\varrho(t)-k}\,dt}{l^{\varrho-k+1}\dfrac{d}{dr}\left(r^{\varrho(r)-k+1}\right)} = \frac{(\varrho-k+1)\; l(lr)^{\varrho(lr)-k}}{l^{\varrho-k+1}\; r^{\varrho(r)-k}\; (r\log r\varrho'(r)+\varrho(r)-k+1)} \sim$$

$$\sim \frac{(\varrho-k+1)\; l^{\varrho-k+1}\; r^{\varrho(r)-k}}{l^{\varrho-k+1}\; r^{\varrho(r)-k}\; (r\log r\varrho'(r)+\varrho(r)-k+1)} \qquad \text{as } r\to\infty,$$

$\to 1$ as $r\to\infty$ since $\varrho(r)\to\varrho$ and $r\log r\varrho'(r)\to 0$ as $r\to\infty$. Hence (β) follows from (1).

Let $k>\varrho+1$. Then $\varrho(r)-k\to\varrho-k<-1$, as $r\to\infty$. Hence $\int_{r_0}^{\infty} t^{\varrho(t)-k}\,dt$ is convergent. So,

$$\int_{lr}^{\infty} t^{\varrho(r)-k}\,dt\to 0 \qquad \text{as } r\to\infty.$$

Also $r^{\varrho(r)-k+1}\to 0$ as $r\to\infty$, since $\varrho+1-k<0$.

Hence, again, by L'Hospital's rule,

$$\text{(2)}\qquad \lim_{r\to\infty}\frac{\displaystyle\int_{lr}^{\infty} t^{\varrho(t)-k}\,dt}{\dfrac{l^{\varrho-k+1}}{k-\varrho-1}\, r^{\varrho(r)-k+1}} = \lim_{r\to\infty}\frac{(k-\varrho-1)\dfrac{d}{dr}\displaystyle\int_{lr}^{\infty} t^{\varrho(t)-k}\,dt}{l^{\varrho-k+1}\dfrac{d}{dr}\left(r^{\varrho(r)-k+1}\right)}$$

provided the latter limit exists.

Now,

$$\frac{d}{dr}\int_{lr}^{\infty} t^{\varrho(t)-k}\,dt = \frac{d}{dr}\left\{\int_{r_0}^{\infty} t^{\varrho(t)-k}\,dt - \int_{r_0}^{lr} t^{\varrho(t)-k}\,dt\right\}$$

$$= -\,l(lr)^{\varrho(lr)-k} \sim -\,l^{\varrho-k+1}\, r^{\varrho(r)-k} \qquad \text{as } r\to\infty.$$

$$\frac{(k-\varrho-1)\dfrac{d}{dr}\displaystyle\int_{lr}^{\infty} t^{\varrho(t)-k}\,dt}{l^{\varrho-k+1}\dfrac{d}{dr}\left(r^{\varrho(r)-k+1}\right)} \sim \frac{-(k-\varrho-1)\; l^{\varrho-k+1}\; r^{\varrho(r)-k}}{l^{\varrho-k+1}\; r^{\varrho(r)-k}\; (r\log r\varrho'(r)+\varrho(r)-k+1)},$$

(as $r\to\infty$) $\to 1$ as $r\to\infty$. Hence (γ) follows from (2).

It is easily seen from the above proof that (β) and (γ) also hold if $\varrho(r)$ is replaced by $\lambda(r)$ and ϱ is replaced by λ.

When $\varrho > 0$, $r^{\varrho(r)}$ is increasing for all r sufficiently large, since it is easily seen that $d/dr(r^{\varrho(r)}) > 0$. However, this may no longer be true when $\varrho = 0$.

2. - An application to entire functions.

We now prove:

Theorem 1. *If f is a non-constant entire function of finite lower order λ and l is a constant $\geqslant 1$, then*

$$\liminf_{r\to\infty} \frac{\log M(lr, f)}{T(r, f)} \leqslant \left\{\frac{\lambda - 1 + \sqrt{1+\lambda^2}}{\lambda + 1 - \sqrt{1+\lambda^2}}\right\} \left\{\frac{\lambda l}{\sqrt{1+\lambda^2} - \lambda}\right\}^{\lambda}, \tag{3}$$

if $\lambda > 0$, and

$$\liminf_{r\to\infty} \frac{\log M(lr, f)}{T(r, f)} = 1, \tag{4}$$

if $\lambda = 0$.

Proof. Since

$$\liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r} = \lambda < \infty,$$

there exists a lower proximate order $\lambda(r)$ relative to $\log M(r, f)$. Let $0 < k < 1$. Then, for $r > 0$,

$$\log M(kr, f) \leqslant \frac{r + kr}{r - kr} T(r, f) = \frac{1+k}{1-k} T(r, f).$$

So,

$$\begin{aligned} T(r, f) &\geqslant \frac{1-k}{1+k} \log M(kr, f) \geqslant \frac{1-k}{1+k} (kr)^{\lambda(kr)} && \text{for } r \geqslant r_0 . \\ &\sim \frac{1-k}{1+k} k^{\lambda} r^{\lambda(r)} && \text{as } r \to \infty, \\ &\sim \frac{1-k}{1+k} k^{\lambda} \frac{1}{l^{\lambda}} (lr)^{\lambda(lr)} && \text{as } r \to \infty, \\ &= \frac{1-k}{1+k} \left(\frac{k}{l}\right)^{\lambda} \log M(lr, f) \end{aligned}$$

for a sequence of values of r tending to ∞.

Hence,

$$\limsup_{r\to\infty} \frac{T(r, f)}{\log M(lr, f)} \geqslant \frac{1-k}{1+k}\left(\frac{k}{l}\right)^{\lambda}. \tag{5}$$

If $\lambda = 0$, this gives

$$\limsup_{r\to\infty} \frac{T(r, f)}{\log M(lr, f)} \geqslant \frac{1-k}{1+k}.$$

Since this holds for all k, $0 < k < 1$, letting $k \to 0$ we obtain

$$\limsup_{r\to\infty} \frac{T(r, f)}{\log M(lr, f)} \geqslant 1 .$$

Since $\limsup_{r\to\infty} (T(r, f)/\log M(lr, f)) \leqslant 1$, because $l \geqslant 1$, we obtain (4). If $\lambda > 0$, then the right member of (5) is a maximum when $k = (\sqrt{1+\lambda^2} - 1)/\lambda$. Substituting this value of k in (5), we obtain (3).

Corollary 1.1. *If f is an entire function of finite order and lower order zero and l is a constant $\geqslant 1$, then*

$$\limsup_{r\to\infty} \frac{T(r, f')}{T(lr, f)} = 1 . \tag{6}$$

Proof. Since f is of finite order, we have

$$\limsup_{r\to\infty} \frac{T(r, f')}{T(lr, f)} \leqslant 1 . \tag{7}$$

Also, the lower order of f' = the lower order of f which equals 0. Hence, by Theorem 1,

$$\liminf_{r\to\infty} \frac{\log M(lr, f')}{T(r, f')} = 1 , \qquad \liminf_{r\to\infty} \frac{\log M(lr, f)}{T(lr, f)} = 1 . \tag{8}$$

Further, since f is of finite order,

$$\log M(r, f) \sim \log M(r, f') , \qquad \text{as } r \to \infty . \tag{9}$$

Hence by (8) and (9),

$$(10)\quad \begin{cases} \limsup\limits_{r\to\infty} \dfrac{T(r,f')}{T(lr,f)} = \limsup\limits_{r\to\infty} \left\{\dfrac{T(r,f')}{\log M(lr,f')}\,\dfrac{\log M(lr,f')}{\log M(lr,f)}\,\dfrac{\log M(lr,f)}{T(lr,f)}\right\} \\ \geqslant \limsup\limits_{r\to\infty} \dfrac{T(r,f')}{\log M(lr,f')}\,\liminf\limits_{r\to\infty}\dfrac{\log M(lr,f')}{\log M(lr,f)}\,\liminf\limits_{r\to\infty}\dfrac{\log M(lr,f)}{T(lr,f)} \\ = 1\,, \end{cases}$$

(6) follows from (7) and (10).

3. - Applications in value distribution theory.

Let $\varphi(r)$ be a positive, non-decreasing function defined for $r\geqslant 1$ and let φ be integrable on $\{1, r\}$ for all $r\geqslant 1$.

Let

$$\limsup_{r\to\infty} \frac{\log \varphi(r)}{\log r} = \varrho < \infty\,.$$

Let l be a constant $>\varrho$ and $Q(r) = r^l\int\limits_r^\infty(\varphi(t)/t^{l+1})\,dt$. We note that $\int\limits_1^\infty(\varphi(t)/t^{l+1})\,dt<\infty$, for, if $\varepsilon>0$ is sufficiently small, then $(\varphi(t)/t^{l+1})\leqslant(t^{\varrho+\varepsilon}/t^{l+1})$ for all t sufficiently large and $l+1-\varrho-\varepsilon>1$.

Let

$$\Phi(r) = \int\limits_1^r \frac{\varphi(t)}{t}\,dt\,.$$

Then $\Phi(r)$ is an increasing function and since φ is increasing, it follows that

$$\limsup_{r\to\infty}\frac{\log \Phi(r)}{\log r} = \limsup_{r\to\infty}\frac{\log\varphi(r)}{\log r} = \varrho\,.$$

Let

$$P(r) = r^l\int\limits_r^\infty \frac{\Phi(t)}{t^{l+1}}\,dt\,.$$

We have the following

Lemma 1. (i)

$$\liminf_{r\to\infty} \frac{Q(r)}{\varphi(r)} \geqslant \frac{1}{l}, \tag{11}$$

$$\liminf_{r\to\infty} \frac{P(r)}{\Phi(r)} \geqslant \frac{1}{l}. \tag{12}$$

(ii) *If* $k > 0$, *then*

$$\liminf_{r\to\infty} \frac{Q(kr)}{\varphi(r)} \leqslant \frac{k^\varrho}{l-\varrho} \tag{13}$$

and

$$\liminf_{r\to\infty} \frac{P(kr)}{\Phi(r)} \leqslant \frac{k^\varrho}{l-\varrho}. \tag{14}$$

(iii)

$$\liminf_{r\to\infty} \frac{Q(r)}{\Phi(r)} \leqslant \frac{\varrho}{l-\varrho} \tag{15}$$

and

$$\liminf_{r\to\infty} \frac{Q(r)}{P(r)} \leqslant \varrho. \tag{16}$$

Proof. (i) Since φ is increasing, we have

$$Q(r) = r^l \int_r^\infty \frac{\varphi(t)}{t^{l+1}}\, dt \geqslant r^l \varphi(r) \int_r^\infty \frac{1}{t^{l+1}}\, dt = \frac{\varphi(r)}{l}.$$

Hence $Q(r)/\varphi(r) \geqslant 1/l$, which proves (11).

The proof of (12) is similar.

(ii) We have

$$\limsup_{r\to\infty} \frac{\log \varphi(r)}{\log r} = \varrho < \infty.$$

Hence there exists a proximate order $\varrho(r)$ relative to $\varphi(r)$. For all r sufficiently large, we then have,

$$Q(kr) = (kr)^l \int_{kr}^{\infty} \frac{\varphi(t)}{t^{k+1}}\, \mathrm{d}t \leqq k^l r^l \int_{kr}^{\infty}{}' t^{\varrho(t)-l-1}\, \mathrm{d}t \sim$$

$$\sim k^l r^l \frac{k^{\varrho-l}\ r^{\varrho(r)-l}}{l-\varrho} \qquad \text{as } r \to \infty, \text{ by } (\gamma)\,.$$

Hence

$$\limsup_{r\to\infty} \frac{Q(kr)}{r^{\varrho(r)}} \leqq \frac{k^\varrho}{l-\varrho}\,. \tag{17}$$

Since $r^{\varrho(r)} = \varphi(r)$ for a sequence of values of r tending to infinity, we obtain (13).

The proof of (14) is similar, for, if $\varrho(r)$ is a proximate order relative to $\Phi(r)$, we obtain, in place of (17),

$$\limsup_{r\to\infty} \frac{P(kr)}{r^{\varrho(r)}} \leqq \frac{k^\varrho}{l-\varrho}\,. \tag{18}$$

(iii)

$$Q(r) = r^l \int_r^{\infty} \frac{\varphi(t)}{t^{l+1}}\, \mathrm{d}t = r^l \int_r^{\infty} \frac{\varphi(t)}{t} \frac{1}{t^l}\, \mathrm{d}t =$$

$$= r^l \left\{ -\frac{\Phi(r)}{r^l} + l \int_r^{\infty} \frac{\Phi(t)}{t^{l+1}}\, \mathrm{d}t \right\},$$

upon integrating by parts, noting that $\Phi(t)/t^l \to 0$ as $t \to \infty$ since $l > \varrho$.

Thus,

$$Q(r) = -\Phi(r) + lP(r)\,. \tag{19}$$

Hence,

$$\liminf_{r\to\infty} \frac{Q(r)}{\Phi(r)} = l \liminf_{r\to\infty} \frac{P(r)}{\Phi(r)} - 1$$

$$\leqq \frac{l}{l-\varrho} - 1\,, \qquad \text{by (14), taking } k = 1\,.$$

$$= \frac{\varrho}{l-\varrho}\,, \qquad \text{which proves (15)}\,.$$

From (19), we have

$$\frac{Q(r)}{P(r)} = l - \frac{\Phi(r)}{P(r)}\,,$$

so,

$$\liminf_{r\to\infty} \frac{Q(r)}{P(r)} = l - \limsup_{r\to\infty} \frac{\Phi(r)}{P(r)}$$

$$\leqslant l - (l-\varrho)\,, \qquad \text{by (14), taking } k = 1\,.$$

$$= \varrho\,, \qquad \text{(which proves (16)}\,.$$

Corollary. *If* $\varrho = 0$ *and* $k \geqslant 1$, *then*

(20) $$\liminf_{r\to\infty} \frac{Q(kr)}{Q(r)} = \frac{1}{l}\,,$$

(21) $$\liminf_{r\to\infty} \frac{P(kr)}{\Phi(r)} = \frac{1}{l}\,,$$

(22) $$\liminf_{r\to\infty} \frac{Q(r)}{\Phi(r)} = 0\,,$$

(23) $$\liminf_{r\to\infty} \frac{Q(r)}{P(r)} = 0\,.$$

Proof. Since φ is non-decreasing and $k \geqslant 1$, we have

$$\frac{Q(kr)}{\varphi(r)} \geqslant \frac{Q(kr)}{\varphi(kr)}\,.$$

Hence

$$\liminf_{r\to\infty} \frac{Q(kr)}{\varphi(r)} \geqslant \liminf_{r\to\infty} \frac{Q(kr)}{\varphi(kr)} \geqslant \frac{1}{l}$$

by (11). (20) now follows from (13), since $\varrho = 0$. (21) follows similarly while (22) and (23) are immediate consequences of (15) and (16) respectively.

In what follows, we denote by $\mathscr{C}$, the set of all (finite) complex numbers and by $\overline{\mathscr{C}}$, the extended complex plane consisting of all (finite) complex numbers and ∞. Thus $\overline{\mathscr{C}} = \mathscr{C} \cup \{\infty\}$.

We now deduce some simple results about entire and meromorphic functions. Let f be a meromorphic function of order 0. Let $n(r) = \sum_{i=1}^{q} n(r, a_i)$, where $a_1, a_2, \ldots, a_q \in \overline{\mathscr{C}}$ and $n(r, a_i) \not\equiv 0$ for at least one $i (1 \leqslant i \leqslant q)$.

Let

$$N(r) = \int_0^r \frac{n(t) - n(0)}{t}\, dt + n(0) \log r .$$

Similarly, let $\bar{n}(r) = \sum_{i=1}^{q} \bar{n}(r, a_i)$, where $\bar{n}(r, a_i) \not\equiv 0$ for at least one i, $1 \leqslant i \leqslant q$ and

$$\bar{N}(r) = \int_0^r \frac{\bar{n}(t) - \bar{n}(0)}{t}\, dt + \bar{n}(0) \log r .$$

as usual, $\bar{n}(r, a) =$ the number of distinct zeros of $f - a$, in $|z| \leqslant r$, for each $a \in \overline{\mathscr{C}}$. Then we have the following

Theorem 2. (i)

$$\liminf_{r\to\infty} \frac{r \int_r^\infty \frac{n(t)}{t^2}\, dt}{N(r)} = 0 , \tag{24}$$

$$\liminf_{r\to\infty} \frac{\int_r^\infty \frac{n(t)}{t^2}\, dt}{\int_r^\infty \frac{N(t)}{t^2}\, dt} = 0 . \tag{25}$$

(ii) *If* $k \geqslant 1$, *then*

$$\liminf_{r\to\infty} \frac{r\displaystyle\int_{kr}^{\infty} \frac{n(t)}{t^2}\,dt}{n(r)} = \frac{1}{k}, \tag{26}$$

$$\liminf_{r\to\infty} \frac{r\displaystyle\int_{kr}^{\infty} \frac{N(t)}{t^2}\,dt}{N(r)} = \frac{1}{k}, \tag{27}$$

with corresponding results obtained by replacing n *by* $\bar{n}$ *and* N *by* $\bar{N}$ *everywhere.*

Note. (24), for an entire function of order 0, is due to LITTLEWOOD. Our proof is different.

Proof. Since f is of order zero,

$$\lim_{r\to\infty} \frac{\log n(r)}{\log r} = \lim_{r\to\infty} \frac{\log \bar{n}(r)}{\log r} = \lim_{r\to\infty} \frac{\log N(r)}{\log r} = \lim_{r\to\infty} \frac{\log \bar{N}(r)}{\log r} = 0\,.$$

Theorem 2 now follows from the preceding corollary, taking $\varphi(r) = n(r)$, and $l = 1$. Similarly, taking $\varphi(r) = \bar{n}(r)$ and $l = 1$, we obtain the corresponding results with $\bar{n}$ and $\bar{N}$ in place of n and N respectively.

Let f be a non-constant meromorphic function of order ϱ and a, b be distinct elements of $\overline{\mathscr{C}}$. Let $n(r) = n(r, a) + n(r, b)$ and

$$N(r) = \int_0^{\infty} \frac{n(t) - n(0)}{t}\,dt + n(0)\log r\,.$$

It is known that, if $0 < \varrho < 1$,

$$\liminf_{r\to\infty} \frac{T(r, f)}{N(r)} \leqq \frac{\Pi\varrho}{\sin \Pi\varrho}$$

and if $\varrho = 0$,

$$\liminf_{r\to\infty} \frac{T(r, f)}{N(r)} \leqq 1\,.$$

We now use the technique of proximate orders to prove

Theorem 3. *Let k be a positive constant.*

(i) *If* $0<\varrho<1$, *then*

$$\liminf_{r\to\infty}\frac{T(kr,f)}{n(r)}\leqslant\frac{\Pi k^{\varrho}}{\sin\Pi\varrho},\tag{18}$$

$$\liminf_{r\to\infty}\frac{T(kr,f)}{N(r)}\leqslant\frac{\Pi\varrho k^{\varrho}}{\sin\Pi\varrho}.\tag{29}$$

(ii) *If* $\varrho=0$, *then*

$$\liminf_{r\to\infty}\frac{T(kr,f)}{N(r)}\leqslant 1.\tag{30}$$

Proof. (i) Let $0<\varrho<1$. Without loss of generality, we may assume that $f(0)=1$, $a=0$ and $b=\infty$. Since $0<\varrho<1$, we then have

$$T(r,f)\leqslant r\int_0^\infty\frac{n(t)}{t(t+r)}\,dt.\tag{31}$$

Again, since $\varrho<1$, f has at most one exceptional value in the sense of Borel and so,

$$\limsup_{r\to\infty}\frac{\log n(r)}{\log r}\geqslant \operatorname{Max}\left\{\limsup_{r\to\infty}\frac{\log n(r,a)}{\log r},\ \limsup_{r\to\infty}\frac{\log n(r,b)}{\log r}\right\}=\varrho.$$

on the other hand,

$$N(2r)\geqslant\int_r^{2r}\frac{n(t)}{t}\,dt\geqslant n(r)\log 2$$

and so,

$$n(r)\leqslant\frac{N(2r)}{\log 2}\leqq\frac{2T(2r,f)+O(1)}{\log 2}$$

hence

$$\limsup_{r\to\infty}\frac{\log n(r)}{\log r}\leqq\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}=\varrho.$$

Thus

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} = \varrho\,.$$

Hence,

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log r} = \varrho\,.$$

Let $\varrho(r)$ be a proximate order relative to $n(r)$ so that $\varrho(r) \to \varrho$ as $r \to \infty$ and $n(r) \leqslant r^{\varrho(r)}$ for $r \geqslant a$ certain r_0. Let δ, ν be positive numbers, with $\delta < \nu$. Then, for r sufficiently large, we have, from (31),

$$(32)\quad \left\{\begin{aligned} T(kr, f) &\leqslant kr \int_0^\infty \frac{n(t)}{t(t+kr)}\,\mathrm{d}t = \\ &= kr \int_0^{r_0} \frac{n(t)}{t(t+kr)}\,\mathrm{d}t + kr \int_{r_0}^{k\delta r} \frac{n(t)}{t(t+kr)}\,\mathrm{d}t + kr \int_{k\delta r}^{k\nu r} \frac{n(t)}{t(t+kr)}\,\mathrm{d}t + \\ &\quad + kr \int_{k\nu r}^{\infty} \frac{n(t)}{t(t+kr)}\,\mathrm{d}t\,. \end{aligned}\right.$$

Now,

$$(33)\quad \left\{\begin{aligned} kr \int_0^{r_0} \frac{n(t)}{t(t+kr)}\,\mathrm{d}t \leqq kr \int_0^{r_0} \frac{n(t)}{tkr}\,\mathrm{d}t = \int_0^{r_0} \frac{n(t)}{t}\,\mathrm{d}t = 0(1) \\ = o(r^{\varrho(r)})\,, \end{aligned}\right.$$

since $\varrho > 0$ and $\varrho(r) \to \varrho$ as $r \to \infty$.

Again,

$$kr \int_{r_0}^{k\delta r} \frac{n(t)}{t(t+kr)}\,\mathrm{d}t \leqq kr \int_{r_0}^{k\delta r} \frac{n(t)}{tkr}\,\mathrm{d}t = \int_{r_0}^{k\delta r} \frac{n(t)}{t}\,\mathrm{d}t \leqslant$$

$$\leqslant \int_{r_0}^{k\delta r} t^{\varrho(t)-1}\,\mathrm{d}t \sim \frac{k^\varrho\,\delta^\varrho\, r^{\varrho(r)}}{\varrho} \qquad \text{as } r \to \infty\,, \text{ by } (\beta)\,.$$

Hence

(34)
$$\limsup_{r\to\infty} \frac{kr\int_{r_0}^{k\delta r} \frac{n(t)}{t(t+kr)}\,dt}{r^{\varrho(r)}} \leqslant \frac{k^{\varrho}\,\delta^{\varrho}}{\varrho}\,.$$

Further,

$$kr\int_{k\delta r}^{k\nu r} \frac{n(t)}{t(t+kr)}\,dt \leqq kr\int_{k\delta r}^{k\nu r} \frac{t^{\varrho(t)}}{t(t+kr)}\,dt = \int_{\delta}^{\nu} \frac{(kru)^{\varrho(kru)}}{u(1+u)}\,du\,, \qquad \text{putting } t = kru,$$

$$\sim \int_{\delta}^{\nu} \frac{(ku)^{\varrho}\, r^{\varrho(r)}}{u(1+u)}\,du \qquad \text{as } r\to\infty \quad \left(\text{since } (kru)^{\varrho(kru)} \sim (ku)^{\varrho}\, r^{\varrho(r)} \text{ as } r\to\infty \text{ uniformly for } \delta\leqslant u\leqslant\nu\right),$$

$$= r^{\varrho(r)}\, k^{\varrho}\int_{\delta}^{\nu} \frac{u^{\varrho-1}}{(1+u)}\,du$$

so,

(35)
$$\limsup_{r\to\infty} \frac{kr\int_{k\delta r}^{k\nu r} \frac{n(t)}{t(t+kr)}\,dt}{r^{\varrho(r)}} \leqslant k^{\varrho}\int_{\delta}^{\nu} \frac{u^{\varrho-1}}{1+u}\,du\,.$$

Finally,

$$kr\int_{k\nu r}^{\infty} \frac{n(t)}{t(t+kr)}\,dt \leqslant kr\int_{k\nu r}^{\infty} \frac{n(t)}{t^2}\,dt \leqslant kr\int_{k\nu r}^{\infty} t^{\varrho(t)-2}\,dt \sim$$

$$\sim \frac{kr(k\nu)^{\varrho-1}\, r^{\varrho(r)-1}}{1-\varrho} \qquad \text{as } r\to\infty, \quad \text{by } (\gamma),$$

$$= \frac{k^{\varrho}\,\nu^{\varrho-1}\, r^{\varrho(r)}}{1-\varrho}\,.$$

Therefore,

(36)
$$\limsup_{r\to\infty} \frac{kr\int_{k\nu r}^{\infty} \frac{n(t)}{t(t+kr)}\,dt}{r^{\varrho(r)}} \leqq \frac{k^{\varrho}\,\nu^{\varrho-1}}{1-\varrho}$$

Using (33), (34), (35) and (36), it follows, from (32), that

$$\limsup_{r\to\infty} \frac{T(kr, f)}{r^{\varrho(r)}} \leqslant \frac{k^{\varrho}\,\delta^{\varrho}}{\varrho} + k^{\varrho}\int_{\delta}^{\nu} \frac{u^{\varrho-1}}{1+u}\,du + \frac{k^{\varrho}\,\nu^{\varrho-1}}{1-\varrho}.$$

Since ν and δ are arbitrary positive numbers, with $\delta<\nu$, letting $\delta\to 0$ and $\nu\to\infty$, we obtain, since $0<\varrho<1$,

$$\limsup_{r\to\infty} \frac{T(kr, f)}{r^{\varrho(r)}} \leqslant k^{\varrho}\int_{0}^{\infty} \frac{u^{\varrho-1}}{1+u}\,du = k^{\varrho}\,\frac{\Pi}{\sin \Pi\varrho}.$$

since $r^{\varrho(r)}=n(r)$ for a sequence of values of r tending to infinity, it follows that

$$\limsup_{r\to\infty} \frac{T(kr, f)}{n(r)} \leqslant \frac{\Pi k^{\varrho}}{\sin \Pi\varrho},$$

which proves (28). From (31), we have

$$T(r, f)\leqslant r\int_{0}^{\infty} \frac{n(t)}{t}\,\frac{1}{t+r}\,dt = r\int_{0}^{\infty} \frac{N(t)}{(t+r)^2}\,dt, \tag{37}$$

since $[N(t)/(t+r)]\to 0$ as $t\to\infty$, for,

$$\limsup_{t\to\infty} \frac{\log N(t)}{\log t} = \varrho < 1.$$

Now, let $\varrho(r)$ be a proximate order relative to $N(r)$ so that $\varrho(r)\to\varrho$ as $r\to\infty$ and $N(r)\leqslant r^{\varrho(r)}$ for $r\geqslant a$ certain r_0. As before, if δ, ν are positive numbers, with $\delta<\nu$, we have, for sufficiently large,

$$\tag{38} \left\{ \begin{aligned} T(kr, f) &\leqslant kr\int_{0}^{\infty} \frac{N(t)}{(t+kr)^2}\,dt \qquad \text{(from (37))},\\ &= kr\int_{0}^{r_0} \frac{N(t)}{(t+kr)^2}\,dt + kr\int_{r_0}^{k\delta r} \frac{N(t)}{(t+kr)^2}\,dt + kr\int_{k\delta r}^{k\nu r} \frac{N(t)}{(t+kr)^2}\,dt + \\ &\qquad + kr\int_{k\nu r}^{\infty} \frac{N(t)}{(t+kr)^2}\,dt. \end{aligned} \right.$$

We have

$$kr\int_0^{r_0}\frac{N(t)}{(t+kr)^2}\,dt \leqq kr\int_0^{r_0}\frac{N(t)}{(kr)^2}\,dt = \frac{1}{kr}\int_0^{r_0}N(t)\,dt = 0(1) \qquad \text{as } r\to\infty\,.$$

Treating the other integrals in (38) as before, we obtain, from (38),

$$\limsup_{r\to\infty}\frac{T(kr,f)}{r^{\varrho(r)}} \leqslant \frac{k^\varrho\,\delta^\varrho}{\varrho} + k^\varrho\int_\delta^\nu\frac{u^\varrho}{(1+u)^2}\,du + \frac{k^\varrho\,\nu^{\varrho-1}}{1-\varrho}\,.$$

Letting $\delta\to 0$ and $\nu\to\infty$, we obtain

$$\limsup_{r\to\infty}\frac{T(kr,f)}{r^{\varrho(r)}} \leqslant k^\varrho\int_0^\infty\frac{u^\varrho}{(1+u)^2}\,du = k^\varrho\,\frac{\Pi\varrho}{\sin\Pi\varrho}\,.$$

Since $r^{\varrho(r)} = N(r)$ for a sequence of values of r tending to infinity, we get

$$\liminf_{r\to\infty}\frac{T(kr,f)}{N(r)} \leqslant k^\varrho\,\frac{\Pi\varrho}{\sin\Pi\varrho}\,,$$

which proves (29).

(ii) Suppose $\varrho = 0$. If f is a rational function, then at least one of $N(r,a)$, $N(r,b)$ is asymptotic to $T(r,f)$ and (30) holds trivially. Suppose f is transcendental. We have, from (31), assuming $f(0)=1$, $a=0$, $b=\infty$,

$$\begin{aligned}
T(kr,f) &\leqslant kr\int_0^\infty\frac{n(t)}{t(t+kr)}\,dt \\
&= kr\int_0^r\frac{n(t)}{t(t+kr)}\,dt + kr\int_r^\infty\frac{n(t)}{t(t+kr)}\,dt \leqslant \\
&\leqslant kr\int_0^r\frac{n(t)}{t(kr)}\,dt + kr\int_r^\infty\frac{n(t)}{t^2}\,dt = \\
&= N(r) + kr\int_r^\infty\frac{n(t)}{t^2}\,dt\,.
\end{aligned}$$

Hence,

$$\liminf_{r\to\infty} \frac{T(kr, f)}{N(r)} \leqslant 1 + k\,, \qquad \liminf \frac{r\displaystyle\int_r^\infty \frac{n(t)}{t^2}\,dt}{N(r)} = 1\,, \qquad \text{by (24)}\,.$$

This proves (30). For an alternative proof of (30), when $k = 1$, using Polya peaks, see Hayman ([1], p. 103).

In what follows, if f is a meromorphic function, we shall denote, by $n(r, a, f)$, the number of zeros of $f - a$ in $|z| \leqslant r$, for each $a \in \overline{\mathscr{C}}$. As usual, if $a = \infty$, by a zero of $f - a$ we mean a pole of $f - a$.

Also

$$N(r, a, f) = \int_0^r \frac{n(t, a, f) - n(O, a, f)}{t}\,dt + n(O, a, f)\log r\,.$$

In a particular context where we are dealing with only one meromorphic function f, we sometimes write $n(r, a)$ and $N(r, a)$ for $n(r, a, f)$ and $N(r, a, f)$ respectively and $n(r, \infty)$ and $N(r, \infty)$ are usually written as $n(r, f)$ and $N(r, f)$ respectively.

Corollary 3.1. *Let f be a meromorphic function of order ϱ and let g_1, g_2 be distinct meromorphic functions such that $T(r, g_i) = o\big(T(r, f)\big)$ as $r \to \infty$ for $i = 1, 2$. Let k be a constant $\geqslant 1$.*

(i) *If* $0 < \varrho < 1$,

(39) $$\liminf_{r\to\infty} \frac{T(kr, f)}{n(r, o, f - g_1) + n(r, o, f - g_2)} \leqslant k^\varrho \frac{\Pi}{\sin \Pi\varrho}$$

(40) $$\liminf_{r\to\infty} \frac{T(kr, f)}{N(r, o, f - g_1) + N(r, o, f - g_2)} \leqslant k^\varrho \frac{\Pi\varrho}{\sin \Pi\varrho}\,.$$

(ii) *If* $\varrho = 0$, *then, for all* $k > 0$,

(41) $$\liminf_{r\to\infty} \frac{T(kr, f)}{N(r, o, f - g_1) + N(r, o, f - g_2)} \leqslant 1\,.$$

Note. g_1 or g_2 may be constant and either g_1 or g_2 may be identically equal to ∞ also.

Proof. Without loss of generality, we may assume that f is transcendental, for, in (i), $\varrho > 0$ and so f is necessarily transcendental, whereas, in (ii), if f were a rational function, then g_1, g_2 would be constants and so (41) would reduce to (30).

Define

$$F(z) = \frac{f(z) - g_1(z)}{f(z) - g_2(z)}.$$

Since $T(r, g_i) = o(T(r, f))$ as $r \to \infty$, $i = 1, 2$, we have $T(r, f - g_i) \sim T(r, f)$ as $r \to \infty$.

Hence

$$T(r, g_2 - g_1) = o(T(r, f))$$

$$= o(T(r, f - g_i)) \qquad \text{as } r \to \infty,\ i = 1, 2.$$

Since

$$F(z) = 1 + \frac{g_2(z) - g_1(z)}{f(z) - g_2(z)},$$

we have

$$T(r, F) = T\left(r, \frac{g_2 - g_1}{f - g_2}\right) + O(1) \sim T\left(r, \frac{1}{f - g_2}\right) \qquad \text{as } r \to \infty,$$

$$\sim T(r, f - g_2) \qquad \text{as } r \to \infty,$$

$$\sim T(r, f) \qquad \text{as } r \to \infty.$$

Thus $T(r, F) \sim T(r, f)$ as $r \to \infty$.

Hence the order of F = the order of f which equals ϱ.

(i) Let $0 < \varrho < 1$. Suppose, first, that $k > 1$. Then, from (28) and (29) applied to F, we have, since $T(kr, F) \sim T(kr, f)$,

$$\liminf_{r \to \infty} \frac{T(kr, f)}{n(r, o, F) + n(r, \infty, F)} \leqslant \frac{\Pi k^{\varrho}}{\sin \Pi\varrho} \tag{42}$$

and

$$\liminf_{r \to \infty} \frac{T(kr, f)}{N(r, o, F) + N(r, \infty, F)} \leqslant \frac{\Pi\varrho k^{\varrho}}{\sin \Pi\varrho}. \tag{43}$$

Now, since $F(z) = (f(z) - g_1(z))/(f(z) - g_2(z))$, we have $n(r, o, F) \leqslant n(r, o, f - g_1) +$

$+ n(r, \infty, g_2)$ and $n(r, \infty, F) \leqslant n(r, o, f - g_2) + n(r, \infty, g_1)$. Hence

$$(44) \qquad n(r, o, F) + n(r, \infty, F) \leqslant n(r, o, f - g_1) + \\ + n(r, f - g_2) + n(r, \infty, g_1) + n(r, \infty, g_2) .$$

Since $k > 1$, $N(kr, \infty, g_1) + N(kr, \infty, g_2) \leqslant \{n(r, \infty, g_1) + n(r, \infty, g_2)\} \cdot \log k$.
Hence

$$n(r, \infty, g_1) + n(r, \infty, g_2) \leqslant \frac{1}{\log k}\{T(kr, g_1) + T(kr, g_2)\} = \\ = o\big(T(kr, f)\big) \qquad \text{as } r \to \infty .$$

Hence, from (44),

$$\limsup_{r\to\infty} \frac{n(r, 0, F) + n(r, \infty, F)}{T(kr, f)} \leqslant \limsup_{r\to\infty} \frac{n(r, o, f - g_1) + n(r, o, f - g_2)}{T(kr, f)} .$$

(39) now follows from (42).
Again, from (44),

$$N(r, o, F) + N(r, \infty, F) \leqslant N(r, o, f - g_1) + N(r, o, f - g_2) + N(r, \infty, g_1) + N(r, \infty, g_2)$$
$$\leqslant N(r, o, f - g_1) + N(r, o, f - g_2) + T(r, g_1) + T(r, g_2)$$
$$= N(r, o, f - g_1) + N(r, o, f - g_2) + o\big(T(kr, f)\big) ,$$

since $k > 1$.
Hence

$$\limsup_{r\to\infty} \frac{N(r, o, F) + N(r, \infty, F)}{T(kr, f)} \leqslant \limsup_{r\to\infty} \frac{N(r, o, f - g_1) + N(r, o, f - g_2)}{T(kr, f)}$$

(40) now follows from (43).
To prove (39) and (40) when $k = 1$, we have, for $k > 1$,

$$\liminf_{r\to\infty} \frac{T(r, f)}{n(r, o, f - g_1) + n(r, o, f - g_2)} \leqslant \liminf_{r\to\infty} \frac{T(kr, f)}{n(r, o, f - g_1) + n(r, o, f - g_2)} \leqslant \\ \leqslant \frac{\Pi k^\varrho}{\sin \Pi \varrho} .$$

Since this holds for all $k > 1$, we have, letting $k \to 1$,

$$\liminf_{r\to\infty} \frac{T(r, f)}{n(r, o, f - g_1) + n(r, o, f - g_2)} \leqslant \frac{\Pi}{\sin \Pi\varrho}.$$

Similarly

$$\liminf_{r\to\infty} \frac{T(r, f)}{N(r, o, f - g_1) + N(r, o, f - g_2)} \leqslant \frac{\Pi\varrho}{\sin \Pi\varrho}$$

(41) follows similarly from (30) applied to F.

Corollary 3.2. *Let f be a meromorphic function of order ϱ.*

(i) *If $0 < \varrho < 1$ and k is a constant $\geqslant 1$, then*

$$\liminf_{r\to\infty} \frac{T(kr, f)}{n(r, o, f - g)} \leqslant \frac{2\Pi k^{\varrho}}{\sin \Pi\varrho} \tag{45}$$

for every meromorphic function g, satisfying $T(r, g) = o(T(r, f))$ as $r \to \infty$, with at most one exception and

$$\liminf_{r\to\infty} \frac{T(kr, f)}{N(r, o, f - g)} \leqslant \frac{2\Pi\varrho k^{\varrho}}{\sin \Pi\varrho} \tag{46}$$

for every meromorphic function g, satilfying $T(r, g) = o(T(r, f))$ as $r \to \infty$, with at most one exception.

(ii) *If $\varrho = 0$ and $k > 0$, then*

$$\liminf_{r\to\infty} \frac{T(kr, f)}{N(r, o, f - g)} \leqslant 2 \tag{47}$$

for every meromorphic function g, satisfying $T(r, g) = o(T(r, f))$ as $r \to \infty$, with at most one exception.

Proof. (i) Let $0 < \varrho < 1$ and $k \geqslant 1$. Suppose there exist two distinct meromorphic functions g_1, g_2, satisfying $T(r, g_i) = o(T(r, f))$ as $r \to \infty$ for $i = 1, 2$, and such that

$$\liminf_{r\to\infty} \frac{T(kr, f)}{n(r, o, f - g_i)} > \frac{2\Pi k^{\varrho}}{\sin \Pi\varrho} \qquad \text{for } i = 1, 2 .$$

Then

$$\limsup_{r\to\infty} \frac{n(r, o, f-g_1) + n(r, o, f-g_2)}{T(kr, f)} \leqslant \limsup_{r\to\infty} \frac{n(r, o, f-g_1)}{T(kr, f)} +$$

$$+ \limsup_{r\to\infty} \frac{n(r, o, f-g_2)}{T(kr, f)} < \frac{\sin \Pi\varrho}{2\Pi k^\varrho} + \frac{\sin \Pi\varrho}{\Pi k^\varrho} = \frac{\sin \Pi\varrho}{\Pi k^\varrho},$$

hence

$$\liminf_{r\to\infty} \frac{T(kr, f)}{n(r, o, f-g_1) + n(r, o, f-g_2)} > \frac{\Pi k^\varrho}{\sin \Pi\varrho},$$

which contradicts (39). This proves (45).

(46) and (47) are proved similarly.

Theorem 4. *Let f be a non-constant entire function of order ϱ and let k be a positive constant. Let $a \in \mathscr{C}$.*

(i) *If $0 < \varrho < 1$, then*

$$\liminf_{r\to\infty} \frac{\log M(kr, f)}{n(r, a, f)} \leqslant \frac{\Pi k^\varrho}{\sin \Pi\varrho},$$

$$\liminf_{r\leftarrow\infty} \frac{\log M(kr, f)}{N(r, a, f)} \leqslant \frac{\Pi\varrho k^\varrho}{\sin \Pi\varrho},$$

(ii) *If $\varrho = 0$, then*

$$\liminf_{r\to\infty} \frac{\log M(kr, f)}{N(r, a, f)} \leqslant 1,$$

(iii) *If $\varrho = 0$ and $k \geqslant 1$, then*

$$\liminf_{r\to\infty} \frac{\log M(kr, f)}{N(r, a, f)} = 1.$$

Proof. The proof of (i) and (ii) is analogous to that of Theorem 3, taking $b = \infty$ and noting that (31) and (37) are valid with $T(r, f)$ replaced by

$\log M(r, f)$. If $k \geqslant 1$, we have

$$N(r, a) \leqslant T(r, f) + O(1) \leqslant T(kr, f) + O(1)$$

$$\leqslant \log M(kr, f) + O(1)$$

and so

$$\limsup_{r \to \infty} \frac{N(r, a)}{\log M(kr, f)} \leqslant 1 .$$

So, (iii) follows from (ii).

Corollary 4.1. *Let f be a non-constant entire function of order ϱ and let k be a positive constant. Let g be any entire function (including a polynomial or a finite constant) such that $\log M(r, g) = o(\log M(r, f))$ as $r \to \infty$*

(i) *If $0 < \varrho < 1$, then*

$$\liminf_{r \to \infty} \frac{\log M(kr, f)}{n(r, o, f - g)} \leqslant \frac{\Pi k^{\varrho}}{\sin \Pi \varrho},$$

$$\liminf_{r \to \infty} \frac{\log M(kr, f)}{N(r, o, f - g)} \leqslant \frac{\Pi \varrho k^{\varrho}}{\sin \Pi \varrho} .$$

(ii) *If $\varrho = 0$, then*

$$\liminf_{r \to \infty} \frac{\log M(kr, f)}{N(r, o, f - g)} \leqslant 1 .$$

(iii) *If $\varrho = 0$, then*

$$\liminf_{r \to \infty} \frac{\log M(kr, f)}{N(r, o, f - g)} = 1 \qquad \text{for } k \geqslant 1 .$$

Proof. Follows by applying Theorem 4 to the entire function $F(z) = f(z) - g(z)$, taking $a = 0$ and noting that $\log M(r, F) \sim \log M(r, f)$.

Theorem 5. *Let f be a meromorphic function ol non-integer order $\varrho > 0$. Let a, b be distinct elements of $\overline{\mathscr{C}}$ and*

$$n(r) = n(r, a, f) + n(r, b, f) , \qquad N(r) = \int_0^r \frac{n(t) - n(0)}{t} \, dt + n(0) \log r .$$

If k is a positive constant, then

$$\liminf_{r\to\infty} \frac{T(kr, f)}{n(r)} \leqslant k^{\varrho} B(\varrho) \tag{47}$$

and

$$\liminf_{r\to\infty} \frac{T(kr, f)}{N(r)} \leqslant \varrho k^{\varrho} B(\varrho) \tag{48}$$

with $B(\varrho) = \big((q+1)A(q)\big)/\big((\varrho - q)(q+1-\varrho)\big)$, *where* $q = [\varrho]$ *and* $A(q) = 1$ *if* $q = 0$; $A(q) = 2(2 + \log q)$ *if* $q \geqslant 1$.

Proof. Without loss of generality, assume that $f(0) = 1$, $a = 0$, $b = \infty$. Since ϱ is not an integer, we have

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} = \limsup_{r\to\infty} \frac{\log N(r)}{\log r} = \varrho\,.$$

We can express $f(z)$ in the form

$$f(z) = \exp\big(Q(z)\big)\frac{P_1(z)}{P_2(z)}\,, \tag{49}$$

where $Q(z)$ is a polynomial of degree $\leqslant q$, $P_1(z) = \prod_{\mu} E(z/a_{\mu}, q)$, and $P_2(z) = \prod_{\nu} E(z/b_{\nu}, q)$ where a_{μ} are the zeros and b_{ν} are the poles of f. Then $\log M(r, \exp[Q(z)]) = O(r^q)$,

$$\log M(r, P_1) \leqslant A(q)\,(q+1)\left\{r^q \int_0^r \frac{n(t, o, f)}{t^{q+1}}\,dt + r^{q+1}\int_r^{\infty} \frac{n(t, o, f)}{t^{q+2}}\,dt\right\},$$

$$\log M(r, P_2) \leqslant A(q)(q+1)\left\{r^q \int_0^r \frac{n(t, \infty, f)}{t^{q+1}}\,dt + r^{q+1}\int_r^{\infty} \frac{n(t, \infty, f)}{t^{q+2}}\,dt\right\}.$$

(See HAYMAN, p. 27, (1.21)).

Hence

$$\left\{\begin{aligned} &\log M(r, \exp[Q(z)]) + \log M(r, P_1) + \log M(r, P_2) \leqslant O(r^q) + \\ &\qquad + A(q)(q+1)\left\{r^q \int_0^r \frac{n(t)}{t^{q+1}}\,dt + q^{q+1}\int_r^{\infty} \frac{n(t)}{t^{q+2}}\,dt\right\}\,. \end{aligned}\right. \tag{50}$$

From (49) and (50), we have,

$$(51)\quad \begin{cases} T(kr, f) \leqslant T(kr, \exp[Q(z)]) + T(kr, P_1) + T(kr, P_2) + O(1) \leqslant \\ \quad \leqslant \log M(kr, \exp[Q(z)]) + \log M(kr, P_1) + \log M(kr, P_2) + O(1) \leqslant \\ \quad \leqslant O(r^q) + A(q)(q+1) \left\{ k^q r^q \int_0^{kr} \frac{n(t)}{t^{q+2}} \mathrm{d}t + k^{q+1} r^{q+1} \int_{kr}^{\infty} \frac{n(t)}{t^{q+2}} \mathrm{d}t \right\} \end{cases}$$

now let $\varrho(r)$ be a proximate order relative to $n(r)$ so that $\varrho(r) \to \varrho$ as $r \to \infty$ and $n(r) \leqslant r^{\varrho(r)}$ for all $r \geqslant r_0$. Since ϱ is a non-integer, we have $q < \varrho < q+1$ and hence $O(r^q) = o(r^{\varrho(r)})$ as $r \to \infty$.

From (51), for r sufficiently large, we have

$$(52)\quad \begin{cases} T(kr, f) \leqslant O(r^q) + A(q)\,(q+1) \left\{ k^q r^q \int_0^{r_0} \frac{n(t)}{t^{q+1}} \mathrm{d}t + k^q r^q \int_{r_0}^{kr} \frac{n(t)}{t^{q+1}} \mathrm{d}t + \right. \\ \qquad\qquad \left. + k^{q+1} r^{q+1} \int_{kr}^{\infty} \frac{n(t)}{t^{q+2}} dt \right\} = \\ = O(r^q) + A(q)\,(q+1) \left\{ k^q r^q \int_{r_0}^{kr} \frac{n(t)}{t^{q+1}} \mathrm{d}t + k^{q+1} r^{q+1} \int_{kr}^{\infty} \frac{n(t)}{t^{q+1}} \mathrm{d}t \right\}. \end{cases}$$

We have

$$k^q r^q \int_{r_0}^{kr} \frac{n(t)}{t^{q+1}} \mathrm{d}t \leqslant k^q r^q \int_{r_0}^{kr} t^{\varrho(t)-q-1} \mathrm{d}t \sim$$

$$\sim k^q r^q \frac{k^{\varrho-q}\, r^{\varrho(r)-q}}{\varrho - q} \qquad \text{as } r \to \infty\text{, by } (\beta)$$

$$= \frac{k^{\varrho}\, r^{\varrho(r)}}{\varrho - q}.$$

So,

$$(53)\qquad \limsup_{r\to\infty} \frac{k^q r^q \int_{r_0}^{kr} \frac{n(t)}{t^{q+1}} \mathrm{d}t}{r^{\varrho(r)}} \leqslant \frac{k^{\varrho}}{\varrho - q}.$$

Also,

$$k^{q+1}\, r^{q+1} \int_{kr}^{\infty} \frac{n(t)}{t^{q+2}}\, dt \leqslant k^{q+1}\, r^{q+1} \int_{kr}^{\infty} t^{\varrho(t)-q-2}\, dt \sim$$

$$\sim k^{q+1}\, r^{q+1} \frac{k^{\varrho-q-1}\, r^{\varrho(r)-q-1}}{q+1-\varrho} \qquad (\text{as } r \to \infty\,,\ \text{by } (\gamma))\,,$$

$$= \frac{k^{\varrho}\, r^{\varrho(r)}}{q+1-\varrho}\,.$$

hence

$$(54) \qquad \limsup_{r\to\infty} \frac{k^{q+1}\, r^{q+1} \int_{kr}^{\infty} \frac{n(t)}{t^{q+2}}\, dt}{r^{\varrho(r)}} \leqslant \frac{k^{\varrho}}{q+1-\varrho}\,.$$

Using (53) and (54), we obtain, from (52),

$$\limsup_{r\to\infty} \frac{T(kr, f)}{r^{\varrho(r)}} \leqslant A(q)(q+1) \left\{\frac{k^{\varrho}}{\varrho - q} + \frac{k^{\varrho}}{q+1-\varrho}\right\} = k^{\varrho} B(\varrho)\,.$$

Since $r^{\varrho(r)} = n(r)$ for a sequence of values of r tending to ∞, this yields (47).

We have

$$r^{q} \int_{0}^{r} \frac{n(t)}{t^{q+1}}\, dt = r^{q} \int_{0}^{r} \frac{n(t)}{t^{q+1}} \cdot \frac{1}{t^{q}}\, dt =$$

$$= r^{q} \left\{\frac{N(r)}{r^{q}} + q \int_{0}^{r} \frac{N(t)}{t^{q+1}}\, dt\right\} \qquad (\text{upon integrating by parts})\,,$$

$$= N(r) + q r^{q} \int_{0}^{r} \frac{N(t)}{t^{q+1}}\, dt\,.$$

Similarly

$$r^{q+1} \int_{r}^{\infty} \frac{n(t)}{t^{q+2}}\, dt = -\, N(r) + (q+1)\, r^{q+1} \int_{r}^{\infty} \frac{N(t)}{t^{q+2}}\, dt\,,$$

noting that $(N(t)/t^{q+1}) \to 0$ as $t \to \infty$, since $\varrho < q+1$. Hence, from (50),

$$(54) \qquad T(r, f) \leqslant O(r^q) + A(q)(q+1)\left\{q\, r^q \int_0^r \frac{N(t)}{t^{q+1}}\, dt + (q+1) r^{q+1} \int_r^\infty \frac{N(t)}{t^{q+2}}\, dt\right\}.$$

Now taking a proximate order $\varrho(r)$ relative to $N(r)$ and proceeding as before, we obtain (48).

Corollary 5.1. *Let f be a meromorphic function of non-integer order $\varrho > 0$ and let g_1, g_2 be meromorphic functions (including constants finite or infinite) such that $T(r, g_i) = o(T(r, f))$ as $r \to \infty$, for $i = 1, 2$. If k is a constant $\geqslant 1$, then*

$$\liminf_{r\to\infty} \frac{T(kr, f)}{n(r, o, f - g_1) + n(r, o, f - g_2)} \leqslant k^\varrho B(\varrho)$$

and

$$\liminf_{r\to\infty} \frac{T(kr, f)}{N(r, o, f - g_1) + N(r, o, f - g_2)} \leqslant \varrho k^\varrho B(\varrho)\,.$$

Proof. Follows by applying Theorem 5 to the function

$$F(z) = \frac{f(z) - g_1(z)}{f(z) - g_2(z)}$$

and using an argument analogous to that of Corollary 3.1.

For an alternative proof of Corollary 5.1 with larger constants in the right members of the inequalities see S. M. Shah [4].

Corollary 5.2. *Let f be a meromorphic function of non-integer order ϱ and $k \geqslant 0$. Then*

$$\text{(i)} \qquad \liminf_{r\to\infty} \frac{T(kr, f)}{n(r, o, f - g)} \leqslant 2k^\varrho B(\varrho)$$

for every meromorphic function g, satisfying $T(r, g) = o(T(r, f))$ as $r \to \infty$, with at most one exception.

$$\text{(ii)} \qquad \liminf_{r\to\infty} \frac{T(kr, f)}{N(r, o, f - g)} \leqslant 2\varrho k^\varrho B(\varrho)\,,$$

for every meromorphic function g, satisfying $T(r, g) = o(T(r, f))$ as $r \to \infty$ with at most one exception.

Theorem 6. *Let f be an entire function of non-integer order $\varrho > 0$ and let $a \in \mathscr{C}$. If k is a positive constant, then*

$$\liminf_{r\to\infty} \frac{\log M(kr, f)}{n(r, a, f)} \leqslant k^{\varrho} B(\varrho) ,$$

$$\liminf_{r\to\infty} \frac{\log M(kr, f)}{N(r, a, f)} \leqslant \varrho k^{\varrho} B(\varrho)$$

where $B(\varrho)$ is as defined in Theorem 5.

Proof. The proof is analogous to that of Theorem 5, taking $b = \infty$ and noting that (51) and (54) are valid with $T(r, f)$ replaced by $\log M(r, f)$.

Corollary 6.1. *Let f be an entire function of non-integer order $\varrho > 0$. If k is a positive constant, then for every entire function g, (including a polynomial or a finite constant), satisfying $\log M(r, g) = o(\log M(r, f))$ as $r \to \infty$,*

$$\liminf_{r\to\infty} \frac{\log M(kr, f)}{n(r, o, f-g)} \leqslant k^{\varrho} B(\varrho)$$

and

$$\liminf_{r\to\infty} \frac{\log M(kr, f)}{N(r, o, f-g)} \leqslant \varrho k^{\varrho} B(\varrho) .$$

Lemma 2. *Let $\varphi(r)$ be a positive function $\geqslant 1$ defined for $r \geqslant 1$ and suppose $\varphi(r)$ is integrable on $\{1, r\}$ for all $r > 1$. Let $\limsup_{r\to\infty} (\log\varphi(r)/\log r) = \varrho < \infty$ and k be a positive constant. Then*

(i)
$$\liminf_{r\to\infty} \frac{\varphi(kr)}{\varphi(r)} \leqslant k^{\varrho} .$$

(ii) *If $\varrho > 0$,*

$$\liminf_{r\to\infty} \frac{\int_1^r \frac{\varphi(kt)}{t}\, dt}{\varphi(r)} \leqslant \frac{k^{\varrho}}{\varrho} .$$

Proof. Let $\varrho(r)$ be a proximate order relative to $\varphi(r)$ so that $\varphi(r) \leqslant r^{\varrho(r)}$ for all $r \geqslant$ a certain r_0 and $\varrho(r) \to \varrho$ as $r \to \infty$. Then, for all r sufficiently large,

$$\varphi(kr) \leqslant (kr)^{\varrho(kr)} \sim k^{\varrho} r^{\varrho(r)} \qquad \text{as } r \to \infty$$

hence

$$\limsup_{r\to\infty} \frac{\varphi(kr)}{r^{\varrho(r)}} \leqslant k^{\varrho} .$$

Since $\varphi(r) = r^{\varrho(r)}$ for a sequence of values of r tending to ∞, this implies (i). Suppose $\varrho > 0$. We have, taking r_0 sufficiently large, for all $r \geqslant r_0$,

$$\int_1^r \frac{\varphi(kt)}{t}\,dt = \int_1^{r_0} \frac{\varphi(kt)}{t}\,dt + \int_{r_0}^r \frac{\varphi(kt)}{t}\,dt \leqslant O(1) + \int_{r_0}^r \frac{(kt)^{\varrho(kt)}}{t}\,dt . \tag{55}$$

Now $(kt)^{\varrho(kt)} \sim k^{\varrho} t^{\varrho(t)}$ as $t \to \infty$. Also since $\int_{r_0}^{\infty} t^{\varrho(t)-1}\,dt$ is divergent it follows that

$$\int_{r_0}^r \frac{(kt)^{\varrho(kt)}}{t}\,dt \sim k^{\varrho} \int_{r_0}^r t^{\varrho(t)-1}\,dt \qquad (\text{as } r\to\infty) , \quad \sim k^{\varrho}\frac{r^{\varrho(r)}}{\varrho} , \qquad (\text{as } r\to\infty) .$$

Hence, from (55),

$$\limsup_{r\to\infty} \frac{\displaystyle\int_1^r \frac{\varphi(kt)}{\varphi(t)}\,dt}{r^{\varrho(r)}} \leqslant \frac{k^{\varrho}}{\varrho} .$$

Since $\varphi(r) = r^{\varrho(r)}$ for a sequence of values of r tending to infinity, this implies (ii).

If f is meromorphic and $a \in \mathbb{C}$, as usual, we denote by $\bar{n}(r, a, f)$ the number of distinct zeros of $f - a$ in $|z| \leqslant r$ and

$$\bar{N}(r, a, f) = \int_0^r \frac{\bar{n}(t, a, f) - \bar{n}(0, a, f)}{t}\,dt + \bar{n}(0, a, f)\log r .$$

Theorem 7. *Let f be a meromorphic function of finite order $\varrho > 0$ and k be a positive constant. Then*

$$\liminf_{r\to\infty} \frac{T(kr, f)}{\bar{n}(r, o, f - g)} \leqslant \frac{3k^{\varrho}}{\varrho} \tag{56}$$

and

$$\liminf_{r\to\infty} \frac{T(kr, f)}{\bar{N}(r, o, f-g)} \leqslant 3k^{\varrho}, \tag{57}$$

for every meromorphic function g, satisfying $T(r, g) = o(T(r, f))$ as $r \to \infty$, with at most two exceptions in each of (56) *and* (57).

Note. (56) and (57) with n and N in place of $\bar{n}$ and $\bar{N}$ respectively are mentioned by S. M. SHAH [4].

Proof. Without loss of generality, assume $f(0) = 1$. Suppose, if possible, that (56) fails to hold for three distinct meromorphic functions g_i, $i = 1, 2, 3$, satisfying $T(r, g_i) = o(T(r, f))$ as $r \to \infty$, $i = 1, 2, 3$. Choose the positive number l such that

$$\limsup_{r\to\infty} \frac{\bar{n}(t, o, f-g_i)}{T(kr, f)} < l < \frac{\varrho}{3k^{\varrho}}, \qquad i = 1, 2, 3. \tag{58}$$

We can then choose r_0 such that

$$\sum_{i=1}^{3} \bar{n}(r, o, f-g_i) < 3l\, T(kr, f) \qquad \text{for all } r \geqslant r_0.$$

Hence

$$\sum_{i=1}^{3} \bar{N}(r, o, f-g_i) = \sum_{i=1}^{3} \int_0^{r_0} \frac{\bar{n}(t, o, f-g_i)}{t}\, dt + \sum_{i=1}^{3} \int_{r_0}^{r} \frac{\bar{n}(t, o, f-g_i)}{t}\, dt \leqslant$$

$$\leqslant O(1) + 3l \int_{r_0}^{r} \frac{T(kt, f)}{t}\, dt.$$

So,

$$\liminf_{r\to\infty} \frac{\sum_{i=0}^{3} \bar{N}(r, o, f-g_i)}{T(r, f)} \leqslant 3l \liminf_{r\to\infty} \frac{\int_{r_0}^{r} \frac{T(kt, f)}{t}\, dt}{T(r, f)}$$

$$\leqslant 3l \frac{k^{\varrho}}{\varrho},$$

by Lemma 2 (ii) taking $\varphi(r) = T(r, f)$.

Hence, from (58),

$$\liminf_{r\to\infty} \frac{\sum_{i=1}^{3} \bar{N}(r, o, f - g_i)}{T(r, f)} < 1 . \tag{59}$$

On the other hand,

$$T(r, f) \leqslant \sum_{i=1}^{3} \bar{N}(r, o, f - g_i) + o\big(T(r, f)\big) \qquad \text{as } r \to \infty,$$

see HAYMAN ([2], p. 47, (2.11)). Hence

$$\liminf_{r\to\infty} \frac{\sum_{i=1}^{3} \bar{N}(r, o, f - g_i)}{T(r, f)} \geqslant 1 \tag{60}$$

which contradicts (59). Thus (56) holds for every g, with at most two exceptions.

Suppose, again, that (57) fails to hold for three distinct functions g_i satisfying $T(r, g_i) = o\big(T(r, f)\big)$ as $r \to \infty$, $i = 1, 2, 3$. Then

$$\limsup_{r\to\infty} \frac{\bar{N}(r, o, f - g_i)}{T(kr, f)} < \frac{1}{3k^{\varrho}} \qquad \text{for } i = 1, 2, 3 .$$

Hence,

$$\limsup_{r\to\infty} \frac{\sum_{i=1}^{3} \bar{N}(r, o, f - g_i)}{T(kr, f)} < \frac{1}{k^{\varrho}} .$$

So,

$$\begin{aligned} \liminf_{r\to\infty} \frac{\sum_{i=1}^{3} \bar{N}(r, o, f - g_i)}{T(r, f)} &\leqslant \limsup_{r\to\infty} \frac{\sum_{i=1}^{3} \bar{N}(r, o, f - g_i)}{T(kr, f)} \liminf_{r\to\infty} \frac{T(kr, f)}{T(r, f)} \\ &< \frac{1}{k^{\varrho}} k^{\varrho} , \qquad \text{by Lemma 2(i)} . \\ &= 1 . \end{aligned}$$

This agains contradicts (60) and so (57) holds for every g, with at most two exceptions.

Corollary 7.1. *Let f be a meromorphic function of finite order $\varrho > 0$. If $k = 1$, then (56) and (57) hold simultaneously for every meromorphic function g, satisfying $T(r, g) = o(T(r, f))$ as $r \to \infty$, with at most two exceptions.*

Proof. We have already seen that each of (56) and (57) holds for every g, with at most two exceptions.

Suppose there exist two distinct meromorphic functions g_1, g_2 satisfying $T(r, g_i) = o(T(r, f))$ *as* $r \to \infty$, such that (56) fails to hold for $g = g_i$, $i = 1, 2$.

Then, as in the proof of Theorem 7, we obtain

$$\liminf_{r\to\infty} \frac{\sum_{i=1}^{3} \bar{N}(r, o, f - g_i)}{T(r, f)} < \frac{2}{3}. \tag{61}$$

We shall show that strict inequality in (57) holds for every g, different from g_1 and g_2. Suppose, on the contrary, that

$$\liminf_{r\to\infty} \frac{T(r, f)}{\bar{N}(r, o, f - g_3)} \geqslant 3 \tag{62}$$

for some function g_3, different from g_1 and g_2, and satisfying $T(r, g_3) = o(T(r, f))$ *as* $r \to \infty$. Then

$$\liminf_{r\to\infty} \frac{\sum_{i=1}^{3} \bar{N}(r, o, f - g_i)}{T(r, f)} \leqslant \liminf_{r\to\infty} \frac{\sum_{i=1}^{2} \bar{N}(r, o, f - g_i)}{T(r, f)} + \limsup_{r\to\infty} \frac{\bar{N}(r, o, f - g_3)}{T(r, f)}$$

$$< 1, \qquad \text{by (61) and (62).}$$

This contradicts (60) and proves our assertion.

Suppose, again, that (57) fails to hold for two distinct meromorphic functions g_1, g_2 satisfying $T(r, g_i) = o(T(r, f))$ *as* $r \to \infty$. Then

$$\limsup_{r\to\infty} \frac{\sum_{i=1}^{2} \bar{N}(r, o, f - g_i)}{T(r, f)} \leqslant \sum_{i=1}^{2} \limsup_{r\to\infty} \frac{\bar{N}(r, o, f - g_i)}{T(r, f)} < \frac{2}{3}. \tag{63}$$

Then strict inequality holds in (56) for every g, different from g_1 and g_2, for, suppose, on the contrary, that

$$\liminf_{r\to\infty} \frac{T(r, f)}{\bar{n}(r, o, f - g_3)} \geqslant \frac{3}{\varrho},$$

for some g_3, different from g_1 and g_2, with $T(r, g_3) = o(T(r, f))$ *as* $r \to \infty$.

Then, as in the proof of Theorem 7,

$$\liminf_{r\to\infty} \frac{\bar{N}(r, o, f - g_3)}{T(r, f)} \leqslant \frac{1}{3}. \tag{64}$$

From (63) and (64) we again obtain $\liminf_{r\to\infty}\{\sum_{i=1}^{3} \bar{N}(r, o, f - g_i)/T(r, f)\} < 1$ which contradicts (60). This completes the proof of the corollary.

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Summary.

In this article, using the properties of proximate orders, and lower proximate orders, some results in the theory of entire and meromorphic functions are obtained. Also, the maximum modulus and the Nevalinna characteristic of entire functions of finite lower order are compared.

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