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Summability (L) of the r^{th} Differentiated Fourier Series. (**)

1. – Suppose that $f(t)$ is LEBESGUE integrable in $(-\pi, \pi)$ and periodic with a period 2π, and let

$$(1.1) \qquad f(t) \sim \tfrac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) .$$

Then the r^{th} differentiated series of (1.1) at $t = x$ is

$$(1.2) \qquad \sum_{n=1}^{\infty} \frac{\mathrm{d}^r}{\mathrm{d}x^r} (a_n \cos nx + b_n \sin nx) .$$

We write

$$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\} ,$$

$$\varphi_r(t) = \tfrac{1}{2}\{f(x+t) + (-1)^r f(x-t)\} ,$$

$$\lambda(t) = \frac{\varphi_r(t)}{t^r} - \frac{C}{r!} ,$$

and

$$\lambda_1(t) = \frac{1}{t} \int_0^t \lambda(u) \mathrm{d}u ,$$

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where C is a function of x.

Definition. The sequence $\{S_n\}$ is said to be summable (L) to the sum S, if, for x in the interval $(0, 1)$,

$$\left(\log \frac{1}{1-x}\right)^{-1} \sum_{n=1}^{\infty} \frac{S_n}{n} x^n ,$$

exists and tends to a finite limit S as $x \to 1-0$. This is simply written as

$$S_n \to S(\mathrm{L}) .$$

(see BORWEIN [**1**]).

2. – This method has been applied by HSIANG [**2**] and consequently by one of the present authors ([**3**], [**4**]) in respect of the summability (L) of the series (1.1) and (1.2) for $r=1$.

As regards the summability (L) of the series (1.2), one of the present authors [**5**] has proved the following theorem:

Theorem. *If, as* $t \to 0$,

$$G(t) = \int_t^{\pi} \frac{g(u)}{u}\, \mathrm{d}u = 0\left(\log \frac{1}{t}\right) ,$$

where

$$g(t) = \frac{\varphi_r(t)}{t^r} - \frac{C}{r!} ,$$

then the series (1.2) *is summable* (L) *to* C.

The object of the present paper is to prove the above theorem with less stringent condition.

Theorem. *If, as* $t \to 0$,

$$X(t) = \int_t^{\pi} \frac{\lambda_1(u)}{u}\, \mathrm{d}u = 0\left(\log \frac{1}{t}\right) ,$$

then the series (1.2) *is summable* (L) *to* C.

The case $r=1$ has been considered by G. DAS and one of the present authors [**6**].

3. - Proof of the Theorem.

Let T_n and S_n be the n^{th} partial sums of the series (1.1) and (1.2) respectively. We have

$$T_n = \frac{1}{\pi}\int_0^\pi \frac{1}{2}\{f(x+t)+f(x-t)\}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt\,.$$

Hence,

$$S_n = \frac{(-1)^r}{\pi}\int_0^\pi \frac{1}{2}\{f(x+t)+(-1)^r f(x-t)\}\frac{d^r}{dt^r}\left\{\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\right\}dt\,.$$

$$= \frac{(-1)^r}{\pi}\int_0^\pi \varphi_r(t)\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\sin nt+\cos nt\right)dt\,.$$

Thus we have,

$$\sum_{n=1}^\infty \frac{S_n}{n}x^n = \frac{(-1)^r}{\pi}\int_0^\pi\left\{\lambda(t)+\frac{C}{r!}\right\}t^r\cdot\frac{d^r}{dt^r}\left\{\operatorname{Cot}\frac{1}{2}t\cdot\sum_1^\infty\frac{\sin nt}{n}x^n+\sum_1^\infty\frac{\cos nt}{n}x^n\,dt\right\}$$

$$= \frac{(-1)^r}{\pi}\int_0^\pi \lambda(t)\cdot t^r\cdot\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)dt\,+$$

$$+\frac{(-1)^{r+1}}{\pi}\int_0^\pi \lambda(t)\cdot t^r\cdot\frac{d^r}{dt^r}\left\{\frac{1}{2}\log(1-2x\cos t+x^2)\right\}dt\,+$$

$$+\frac{(-1)^r}{\pi}\int_0^\pi t^r\cdot\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)dt\,+$$

$$+\frac{(-1)^{r+1}}{\pi}\int_0^\pi t^r\cdot\frac{d^r}{dt^r}\left\{\frac{1}{2}\log(1-2x\cos t+x^2)\right\}dt\,,$$

$$= P_1 + P_2 + \frac{C}{r!} P_3 + \frac{C}{4!} P_4 . \tag{3.1}$$

In order to consider P_1 we require the following estimates:

$$\text{(3.2) } (i) \qquad \left| t^{r+2} \cdot \frac{d^{r+2}}{dt^{r+2}} \operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right| \leqslant \frac{M}{1-x} , \; (t < 1 - x) ,$$

where M is independent of x.

$$\text{(3.3) (ii)} \qquad \left| t^{r+2} \cdot \frac{d^{r+2}}{dt^{r+2}} \left(\operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right) \right| \leqslant \frac{M_1 (1-x)}{t^2} , \; (t > 1 - x) ,$$

where M_1 is independent of x.
The proof of these is similar to that of (3.4) and (3.5) of [**5**].

Now

$$P_1 = \frac{(-1)^r}{\pi} \int_0^\pi \{ t \lambda_1'(t) + \lambda_1(t) \} \cdot t^r \cdot \frac{d^r}{dt^r} \left(\operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right) dt ,$$

$$= \frac{(-1)^r}{\pi} \int_0^\pi \lambda_1(t) \cdot t^r \cdot \frac{d^r}{dt^r} \left(\operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t}\, dt \right) +$$

$$+ \frac{(-1)^r}{\pi} \int_0^\pi \lambda_1'(t) \cdot t^{r+1} \cdot \frac{d^r}{dt^r} \left(\operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right) dt ,$$

$$= \frac{(-1)^r}{\pi} (P_{11} + P_{12}) , \text{ say.} \tag{3.4}$$

Then,

$$P_{11} = - \int_0^\pi t X'(t) \cdot t^r \cdot \frac{d^r}{dt^r} \left(\operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right) dt ,$$

$$= - \left[X(t) \cdot t^{r+1} \cdot \frac{d^r}{dt^r} \left(\operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right) \right]_0^\pi +$$

$$+ \int_0^\pi X(t) \cdot \frac{d}{dt} \left\{ t^{r+1} \cdot \frac{d^r}{dt^r} \left(\operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right) \right\} dt ,$$

$$= \int_0^\pi X(t) \cdot \frac{d}{dt} \left\{ t^{r+1} \cdot \frac{d^r}{dt^r} \left(\operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \operatorname{Cos} t} \right) \right\} dt ,$$

$$= \left(\int_0^{1-x} + \int_{1-x}^\pi \right) X(t) \cdot \frac{d}{dt} \left\{ t^{r+1} \cdot \frac{d^r}{dt^r} \left(\operatorname{Cot} \frac{1}{2}\, t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right) \right\} dt .$$

But,

$$\int_0^{1-x} X(t)\cdot\frac{d}{dt}\left\{t^{r+1}\cdot\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)\right\}dt = 0\left(\log\frac{1}{1-x}\right)$$

and

$$\int_{1-x}^{\pi} X(t)\cdot\frac{d}{dt}\left\{t^{r+1}\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)\right\}dt = 0\left(\log\frac{1}{1-x}\right)$$

as proved in ([**5**], p. 19), (3.7) and (3.8) replacing $X(t)$ in place of $G(t)$.

We therefore have,

$$P_{11} = 0\left(\log\frac{1}{1-x}\right). \tag{3.5}$$

And again,

$$\begin{aligned}
P_{12} &= \int_0^{\pi} \lambda_1'(t)\cdot t^{r+1}\cdot\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)dt, \\
&= \left[\lambda_1(t)\cdot t^{r+1}\cdot\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)\right]_0^{\pi} \\
&\quad - \int_0^{\pi}\lambda_1(t)\cdot\frac{d}{dt}\left\{t^{r+1}\cdot\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)\right\}dt, \\
&= \int_0^{\pi} tX'(t)\cdot\frac{d}{dt}\left\{t^{r+1}\cdot\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)\right\}dt, \\
&= \left[X(t)\cdot t\cdot\frac{d}{dt}\left\{t^{r+1}\cdot\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)\right\}\right]_0^{\pi} - \\
&\quad - \int_0^{\pi} X(t)\frac{d}{dt}\left[t\frac{d}{dt}\left\{t^{r+1}\frac{d^r}{dt^r}\left(\operatorname{Cot}\frac{1}{2}t\cdot\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)\right\}\right]dt, \\
&= -\left(\int_0^{1-x} + \int_{1-x}^{\pi}\right),
\end{aligned}$$

$$= -(P_{121} + P_{122}), \quad \text{say.} \tag{3.6}$$

Now

$$P_{121} = (r+1)^2 \int_0^{1-x} X(t) \cdot t^r \cdot \frac{d^r}{dt^r}\left(\operatorname{Cot} \frac{1}{2} t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t}\right) dt +$$

$$+ (2r+3) \int_0^{1-x} X(t) \cdot t^{r+1} \cdot \frac{d^{r+1}}{dt^{r+1}}\left(\operatorname{Cot} \frac{1}{2} t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t}\right) dt +$$

$$+ \int_0^{1-x} X(t) \cdot t^{r+2} \cdot \frac{d^{r+2}}{dt^{r+2}}\left(\operatorname{Cot} \frac{1}{2} t \cdot \tan^{-1} \frac{x \sin t}{1 - x \cos t}\right) dt .$$

(on simple differentiation)

$$= (r+1)^2 P_{1211} + (2r+3) P_{1212} + P_{1213}, \quad \text{say.} \tag{3.7}$$

As in ([**5**], (3.7), p. 19), replacing $G(t)$ by $X(t)$ we have, each of

$$P_{1211} \text{ and } P_{1212} = 0\left(\log \frac{1}{1-x}\right), \tag{3.8}$$

and using (3.2) under similar conditions, we have

$$P_{1213} = 0\left(\log \frac{1}{1-x}\right). \tag{3.9}$$

Hence by (3.7), (3.8) and (3.9),

$$P_{121} = 0\left(\log \frac{1}{1-x}\right). \tag{3.10}$$

Also, dealing P_{122} in the same way as P_{121}, using the estimate (3.3) above, exactly in the same manner, it is proved that,

$$P_{122} = 0\left(\log \frac{1}{1-x}\right). \tag{3.11}$$

Thus, by (3.4), (3.5), (3.6), (3.10) and (3.11),

$$P_1 = 0\left(\log \frac{1}{1-x}\right). \tag{3.12}$$

To consider P_2 we require again the following estimates:

(3.13) (iii) $$\left| \frac{d^{r+2}}{dt^{r+2}} \left\{ \frac{1}{2} \log (1 - 2x \operatorname{Cos} t + x^2) \right\} \right| \leqslant \frac{M_2}{(1-x)^{r+2}}, \quad (t < 1 - x),$$

where M_2 is independent of x.

(3.14) (iv) $$\left| \frac{d^{r+2}}{dt^{r+2}} \left\{ \frac{1}{2} \log (1 - 2x \operatorname{Cos} t + x^2) \right\} \right| \leqslant \frac{M_3}{t^{r+2}}, \quad (t > 1 - x),$$

where M_3 is independent of x.

The proof is similar to that of (3.9) and (3.10) of ([5], p. 20).

Now,

$$P_2 = \frac{(-1)^{r+1}}{\pi} \int_0^\pi t^r \{t \lambda_1'(t) + \lambda_1(t)\} \cdot \frac{d^r}{dt^r} \left\{ \frac{1}{2} \log (1 - 2x \cos t + x^2) \right\} dt,$$

$$= \frac{(-1)^{r+1}}{\pi} \int_0^\pi \lambda_1(t) \cdot t^r \cdot \frac{d^r}{dt^r} \left\{ \frac{1}{2} \log (1 - 2x \cos t + x^2) \right\} dt +$$

$$+ \frac{(-1)^{r+1}}{\pi} \int_0^\pi \lambda_1'(t) \cdot t^{r+1} \cdot \frac{d^r}{dt^r} \left\{ \frac{1}{2} \log (1 - 2x \cos t + x^2) \right\} dt,$$

(3.15) $$= \frac{(-1)^{r+1}}{\pi} (P_{21} + P_{22}), \text{ say.}$$

Proceeding exactly in the same way as P_{11}, we have,

$$P_{21} = -\left(\int_0^{1-x} + \int_{1-x}^\pi \right) \chi(t) \cdot \frac{d}{dt} \left\{ t^{r+1} \cdot \frac{d^r}{dt^r} \left(\frac{1}{2} \log (1 - 2x \cos t + x^2) \right) \right\} dt$$

and as proved in ([5], (3.11)), we have,

(3.16) $$P_{21} = 0 \left\{ (1 - x) \log \frac{1}{(1-x)} \right\}$$

Proceeding as before, we have also,

$$P_{22} = \int_0^\pi \lambda_1'(t) \cdot t^{r+1} \cdot \frac{d^r}{dt^r} \left\{ \frac{1}{2} \log (1 - 2x \cos t + x^2) \right\} dt,$$

$$= \left[\lambda_1(t) \cdot t^{r+1} \cdot \frac{d^r}{dt^r} \left\{ \frac{1}{2} \log (1 - 2x \cos t + x^2) \right\} \right]_0^\pi -$$

$$- \int_0^\pi \lambda_1(t) \cdot \frac{d}{dt} \left[t^{r+1} \cdot \frac{d^r}{dt^r} \left\{ \frac{1}{2} \log (1 - 2x \cos t + x^2) \right\} dt,\right.$$

$$= O(1) + \int_0^\pi t\chi'(t) \cdot \frac{d}{dt} \left[t^{r+1} \cdot \frac{d^r}{dt^r} \left\{ \frac{1}{2} \log (1 - 2x \cos t + x^2) \right\} \right] dt,$$

$$= 0 \left(\log \frac{1}{1-x} \right) + \left[\chi(t) \cdot t \cdot \frac{d}{dt} \left\{ t^{r+1} \cdot \right.\right.$$

$$\left.\left. \cdot \frac{d^r}{dt^r} \left(\frac{1}{2} \log (1 - 2x \cos t + x^2) \right) \right\} \right]_0^\pi -$$

$$- \int_0^\pi \chi(t) \cdot \frac{d}{dt} \left[t \cdot \frac{d}{dt} \left\{ t^{r+1} \cdot \frac{d^r}{dt^r} \left(\frac{1}{2} \log (1 - 2x \cos t + x^2) \right) \right\} \right] dt,$$

$$= - \left(\int_0^{1-x} + \int_{1-x}^\pi \right) \chi(t) \cdot \frac{d}{dt} \left[t \cdot \frac{d}{dt} \left\{ t^{r+1} \cdot \right.\right.$$

$$\left.\left. \cdot \frac{d^r}{dt^r} \left(\frac{1}{2} \log (1 - 2x \cos t + x^2) \right) \right\} \right] dt,$$

$$(3.17) \qquad = - (P_{221} + P_{222}),$$

say.

Now, to deal with P_2 in the similar way as before, we have,

$$P_{221} = \int_0^{1-x} \chi(t) \cdot \frac{d}{dt}\left[t \cdot \frac{d}{dt}\left\{t^{r+1} \cdot \frac{d^r}{dt^r}\left(\frac{1}{2}\log(1-2x\cos t + x^2)\right)\right\}\right] dt,$$

$$= (r+1)^2 \int_0^{1-x} \chi(t) \cdot t^r \cdot \frac{d^r}{dt^r}\left\{\frac{1}{2}\log(1-2x\cos t + x^2)\right\} dt +$$

$$+ (2r+3) \int_0^{1-x} \chi(t) \cdot t^{r+1} \cdot \frac{d^{r+1}}{dt^{r+1}}\left\{\frac{1}{2}\log(1-2x\cos t + x^2)\right\} dt +$$

$$+ \int_0^{1-x} \chi(t) \cdot t^{r+2} \cdot \frac{d^{r+2}}{dt^{r+2}}\left\{\frac{1}{2}\log(1-2x\cos t + x^2)\right\} dt.$$

And again, taking into consideration the estimate (3.13), we have, as in [**5**], (3.11),

$$(3.18) \qquad P_{221} = 0\left\{(1-x)\log\frac{1}{1-x}\right\}.$$

Exactly on similar consideration, taking the help of estimate (3.14), it is easily proved that

$$(3.19) \qquad P_{222} = 0\left\{(1-x)\log\frac{1}{1-x}\right\}.$$

Thus by (3.15), (3.16), (3.17), (3.18), (2.19) we have,

$$(3.20) \qquad P_2 = 0\left\{(1-x)\log\frac{1}{1-x}\right\}.$$

The consideration of P_3 and P_4 are exactly similar to those of P_3 and P_4 of [**5**] and therefore following the same proof, we have,

$$P_4 = 0\left(\log\frac{1}{1-x}\right)$$

and then finally,

$$\frac{1}{\log(1-x)^{-1}} \frac{CP_3}{r!} \to C \text{ as } x \to 1-0.$$

This completes the proof of the theorem.

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