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# Problem of plane strain in a transversely isotropic non-homogeneous Composite circular cylinder. (**)

## 1. - Introduction.

The materials like reinforced plastic, whisker and fibre-reinforced are very useful in the design of modern structures. They are such that their YOUNG' s modulus and POISSON's ratio differ in different directions. Although the solutions of problems are numerious for materials whose elastic coefficients are same at all points within the body in question, there are materials where these vary considerably from point to point. The works of OLSZAK [1] show that the recent trend of research work concern non-homogeneous elastic material of both isotropic and non-isotropic. Recently CHAKRAVORTY and BHANJA [2] have solved some problems of plane strain in transversely isotropic non-homogeneous cylindrical shell assuming that the elastic coefficients and the density of the material vary exponentially. In the present problem the outer material has been to be transversely isotropic non-homogeneous and the inner material homogeneous, the density throughout being constant.

## 2. - Formulation of the problem.

We assume that the material possesses an axis of symmetry in the sense that its elastic behaviour in all directions at right angles to axis is the same. We introduce the cylindrical coordinates $r$, $\theta$, $z$ with the $Z$-axis along the axis of symmetry of material.

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In the plane strain problem symmetrical about the $Z$-axis [**3**], $u$ is a function of $r$ only, $v = 0$ and $w = ez$, where $e$ is the extension (constant) along the $Z$-axis and $u$, $v$, $w$ are the components of displacement in the directions of $r, \theta, z$ respectively due to the rotation of the cylinder with the constant angular velocity $\Omega$ about the $Z$-axis. The stress components are given by [**3**]

$$(1)\qquad \begin{cases} \sigma_r = a_{11}\dfrac{du}{dr} + a_{12}\dfrac{u}{r} + a_{13}e\,, \\[2ex] \sigma_\theta = a_{12}\dfrac{du}{dr} + a_{11}\dfrac{u}{r} + a_{13}\,e\,, \\[2ex] \sigma_z = a_{13}\dfrac{du}{dr} + a_{13}\dfrac{u}{r} + a_{33}e\,, \\[2ex] \tau_{\theta z} = \tau_{rz} = \tau_{r\theta} = 0\,, \end{cases}$$

where the elastic coefficients $a_{ij}$'s are functions of $r$ and not more constant.

Two of the equations of equilibrium [**3**] are identically satisfied by (1) while the third reduces to

$$(2)\qquad \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \varrho\Omega^2 r = 0\,,$$

for rotation about the $Z$-axis with the constant angular velocity $\Omega$ where $\varrho$ is the constant density of the material.

Henceforth we shall use the superscripts $o$ and $i$ to indicate the corresponding displacement and stress components in the outer $a \leqslant r \leqslant b$ and the inner $o \leqslant r \leqslant a$ regions respectively.

(*a*) *Analysis of the outer material.*

Let $a$ and $b$ be the inner and outer radii of the cylindrical shell of non-homogeneous transversely isotropic material. We characterise the non-homogeneity of the material within $a \leqslant r \leqslant b$ by

$$(3)\qquad a_{ij} = c_{ij}\exp(-k\cdot r^m)\,, \qquad (k \neq 0)$$

where $c_{ij}$ are constants and $m$ is a rational number.

By substituting (1) and (3) in (2) we obtain

$$(4)\quad \begin{cases} \dfrac{d^2u^0}{dr^2} + \left[\dfrac{1}{r} - mkr^{m-1}\right]\dfrac{du^0}{dr} - \left[\dfrac{1}{r^2} + \dfrac{mkc_{12}}{c_{11}}r^{m-2}\right]u^0 = \\ \qquad = \dfrac{c_{13}}{c_{11}}mke_1r^{m-1} - \dfrac{\varrho\Omega^2}{c_{11}}r\exp(kr^m)\,, \end{cases}$$

where $e_1$ stands for $e$ in $a \leqslant r \leqslant b$.

The complementary solution of (4) will satisfy the equation

$$(5)\quad \frac{d^2u^0}{dr^2} + \left[\frac{1}{r} - mkr^{m-1}\right]\frac{du^0}{dr} - \left[\frac{1}{r^2} + \frac{mkc_{12}}{c_{11}}r^{m-2}\right]u^0 = 0\,.$$

By substituting $u^0 = v\exp(x/2)$; $x = kr^m$ and $2c_{12} = mc_{11}$ in (5) we get

$$(6)\quad \frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} - \left(\frac{1}{4} + \frac{1}{m^2x^2}\right)v = 0\,,$$

from which we obtain

$$(7)\quad v = A_1I_{1/m}(x/2) + B_1K_{1/m}(x/2)$$

where $I$ and $K$ are the modified BESSEL functions of first and second kind respectively and $A_1$ and $B_1$ are constants. If $(1/m) - (1/2)$ is an integer the solution (7) may be written in the closed form. For example if $m = 2/3$ that is if $3c_{12} = c_{11}$ we have after some reduction [4]

$$(8)\quad u^0 = [A_2(2-x)\exp(x) + B_2(2+x)]x^{-3/2}$$

where $A_2$ and $B_2$ are constants.

Other solutions in the closed form are possible for $m = (2/5)$, $c_{12}/c_{11} = (1/5)$; $m = (2/7)$, $(c_{12}/c_{11}) = (1/7)$; and so on. But these become more complicated, in addition to which the value of $c_{12}/c_{11}$ becomes progressively less realistic for the material under investigation. We confine our investigation to (8) and by the method of variation of parameters arrive at the complete solution of (4) as

$$(9)\quad \begin{cases} u^0 = A[2r^{-1} - kr^{-1/3}]\exp(kr^{2/3}) + B[2r^{-1} + kr^{-1/3}] - \dfrac{c_{13}e_1r}{4c_{12}} - \\ - \dfrac{\varrho\Omega^2}{16k^3c_{12}}\cdot[20r - 10kr^{5/3} + 3k^2r^{7/3}]\exp(kr^{2/3})\,, \end{cases}$$

where $A$ and $B$ are constants.

Consequently the non-vanishing components of stress are

$$(10)\quad \begin{cases} \sigma_r^0 = \dfrac{4c_{12}}{r^2}\left[A\left(k r^{2/3} - \dfrac{k^2}{2} r^{4/3} - 1\right) - B \exp\left(-k r^{2/3}\right)\right] \\ \quad - \dfrac{\varrho\Omega^2}{8\,k^2}\left[40 - 10kr^{2/3} + 2\,k^2 r^{4/3} + 3\,k^3 r^2\right], \\ \sigma_\theta^0 = \dfrac{4c_{12}}{r^2}\left[A\left(1 - \dfrac{k}{3} r^{2/3} - \dfrac{k^2}{6} r^{4/3}\right) + B\left(1 + \dfrac{2}{3}\,k\,r^{2/3}\right)\exp\left(-kr^2\right)\right] \\ \quad - \dfrac{\varrho\Omega^2}{24\,K}\left[120 - 50\,kr^{2/3} + 14\,k^2 r^{4/3} + 3\,k^3 r^2\right], \\ \sigma_z^0 = \dfrac{2\,kc_{13}}{3r^{4/3}}\left[A(1 - kr^{2/3}) + B\exp\left(-kr^{2/3}\right)\right] \\ \quad - \dfrac{\varrho\Omega^2 c_{13}}{8K^3\,c_{12}}\left[20 - \dfrac{20K}{3} r^{2/3} + \dfrac{5K^2}{3} r^{4/3} + K^3 r^2\right] \\ \quad + \dfrac{e_1}{2c_{12}}\left[2c_{12}c_{33} - c_{13}^2\right]\exp(-kr^{2/3}),\,. \end{cases}$$

(*b*) *Analysis of the inner material*

For the inner homogeneous material ($0 \leqslant r \leqslant a$) we have $a_{ij} = c_{ij}$. Assuming that $3c_{12} = c_{11}$ we get the only equation of equilibrium to be satisfied by $u^i$ as

$$(11)\qquad \frac{d^2u^i}{dr^2} + \frac{1}{r}\frac{du^i}{dr} - \frac{u^i}{r^2} + \frac{\varrho\Omega^2 r}{c_{11}} = 0\,.$$

The solution of (11) appropriate to the problem is

$$(12)\qquad u^i = Cr - \frac{\varrho\Omega^2 r^3}{24\,c_{12}}$$

where $C$ is a constant. Then the non-vanishing stress-components are

$$(13)\quad \begin{cases} \sigma_r^i = 4\,C\,c_{12} + e_2 c_{13} - \dfrac{5\varrho\Omega^2 r^2}{12}\,, \\ \sigma_\theta^i = 4\,C\,c_{12} + e_2 c_{13} - \dfrac{\varrho\Omega^2 r^2}{4}\,, \\ \sigma_r^i = 2\,C\,c_{13} + e_2 c_{33} - \dfrac{\varrho\Omega^2 c_{13} r^2}{6c_{12}} \end{cases}$$

where $e_2$ is the value of $e$ in $0 \leqslant r \leqslant a$.

(c) *Determination of constants A, B, C.*

The boundary and interface conditions are

$$(14)\quad \begin{cases} \text{(i)} & u_0 = u^i, \quad w^0 = w^i; \quad \sigma_r^0 = \sigma_r^i \quad \text{when } r = a\,, \\ \text{(ii)} & \sigma_r^0 = 0\,, \quad \text{when} \quad r = b\,, \\ \text{(iii)} & \int_0^a \sigma_z^i r\,\mathrm{d}r + \int_a^0 \sigma_z^0 r\,\mathrm{d}r = 0\,. \end{cases}$$

From the condition $w^0 = w^i$ when $r = a$ we get

$$(15)\qquad e_1 = e_2 = e \qquad \text{(say)}.$$

And from the conditions of (14) we obtain

$$(16)\quad \begin{cases} A = \dfrac{1}{\Delta}\,[b_1\Delta_{11} + b_2\Delta_{21} + b_3\Delta_{31} + b_4\Delta_{41}]\,, \\ B = \dfrac{1}{\Delta}\,[b_1\Delta_{12} + b_2\Delta_{22} + b_3\Delta_{32} + b_4\Delta_{42}]\,, \\ C = \dfrac{1}{\Delta}\,[b_1\Delta_{13} + b_2\Delta_{23} + b_3\Delta_{33} + b_4\Delta_{43}]\,, \\ e = \dfrac{1}{\Delta}\,[b_1\Delta_{14} + b_2\Delta_{24} + b_3\Delta_{34} + b_4\Delta_{44}]\,, \end{cases}$$

where $\Delta_{ij}$ is the cofactor of $d_{ij}$ in the determinant

$$(17)\quad \left\{ \Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{vmatrix}\,, \right.$$

where

$$(18)\quad \begin{cases} d_{11} = (y^2 - 2y + 2)\,; \quad d_{12} = 2\exp(-y)\,; \quad d_{13} = d_{14} = 0\,, \\ d_{21} = (x^2 - 2x + 2)\,; \quad d_{22} = 2\exp(-x)\,; \quad d_{23} = 2a^2\,;\ d_{24} = \dfrac{a^2 c_{13}}{2c_{12}}\,, \\ d_{31} = (2 - x)\exp(x)\,;\ d_{32} = (2 + x)\,;\ d_{33} - d_{33} = -a^2\,;\ d_{34} = \dfrac{-a^2 c_{13}}{4c_{12}} \\ d_{41} = (x - y)(x + y - 2)\,; \quad d_{42} = 2(\exp(-x) - \exp(-y))\,; \quad d_{43} = 2a^2\,; \\ d_{44} = a^2\left[\dfrac{c_{33}}{c_{13}} + \dfrac{3}{2x^2}\left(\dfrac{2c_{33}}{c_{13}} - \dfrac{c_{13}}{c_{12}}\right)\{h(x) - h(y)\}\right]; \end{cases}$$

$$(19)\quad \begin{cases} b_1 = -[3y^3 + 2y^2 - 10y + 40](b/a)^2\,\dfrac{D}{2x^3}\,, \\ b_2 = \left[\dfrac{5}{3} - (3\,x^3 + 2\,x - 10x + 40)\,\dfrac{1}{2x^3}\right]\cdot D, \\ b_3 = \left[(3\,x^2 - 10x + 20)\exp(x)\cdot\dfrac{1}{2x^3} - \dfrac{1}{3}\right]\cdot D\,, \\ b_4 = \left[\dfrac{2}{3} + \{G(y) - G(x)\}\,\dfrac{1}{2x^6}\right]\cdot D\,, \end{cases}$$

where

$$D = \frac{\varrho\Omega^2 a^4}{8c_{12}}\,; \qquad h(t) = (t^2 + 2t + 2)\exp(-t)\,,$$

$$G(t) = (t^3 + 2t^2 - 10t + 40)t^3\,; \qquad x = ka^{2/3} = \lambda \qquad \text{and} \qquad y = \lambda(b/a)^{2/3}\,.$$

### 3. - Numerical discussion.

For computing the numerical values we assume $b/a = 2$; $C_{33}/C_{12} = 2.46$ and $C_{13}/C_{12} = 0.69$ (as in BERYL). We use the symbols

$$P = \frac{\sigma_r}{\varrho\Omega^2 a^2}\,; \qquad Q = \frac{\sigma_\theta}{\varrho\Omega^2 a^2}\,; \qquad \bar{Q} = \frac{\sigma_\theta^i - \sigma_\theta^0}{\varrho\Omega^2 a^2}\,; \qquad E = -\frac{8c_{12}\,e}{\varrho\Omega^2 a^2}\,;\ \delta = \frac{r}{a}$$

and $P_H$ and $Q_H$ for the respective values of $P$ and $Q$ when the complete cylinder is made of homogeneous material.

TABLE 1 (Values of $P$ and $Q$ for $\lambda = 1$)

| $\delta$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $P$ | 2.0204 | 2.0038 | 1.9413 | 1.8704 | 1.7538 | 1.5037 |
| $Q$ | 1.7704 | 1.7604 | 1.7304 | 1.6804 | 1.6104 | 1.5204 |
| $P_H$ | 1.6667 | 1.6500 | 1.6000 | 1.5164 | 1.4000 | 1.2500 |
| $Q_H$ | 1.6667 | 1.6567 | 1.6067 | 1.5767 | 1.5067 | 1.4167 |

TABLE 2 (Values of $P$ and $Q$ for $\lambda = 1$)

| $\delta$ | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|---|---|---|---|---|---|---|
| $P$ | 1.5037 | 1.2953 | 0.9965 | 0.6854 | 0.3542 | 0.0000 |
| $Q$ | 0.9906 | 0.9398 | 0.8330 | 0.6862 | 0.5156 | 0.3296 |
| $P_H$ | 1.2500 | 1.0667 | 0.9333 | 0.6000 | 0.3167 | 0.0000 |
| $Q_H$ | 1.4167 | 1.3067 | 1.1600 | 1.0267 | 0.8567 | 0.6667 |

TABLE 3 (Values of $\bar{Q}$ on $\delta = 1$)

| $\lambda$ | 1 | 2 | 3 |
|---|---|---|---|
| $\bar{Q}$ | 0.5298 | 1.1567 | 1.4873 |

TABLE 4 (Values of $E$)

| $\lambda$ | 1 | 2 | 3 |
|---|---|---|---|
| $E$ | 2.7628 | 4.0767 | 4.9814 |

The values of $P_H$ and $Q_H$ and $P$ and $Q$ are given in the Table 1 when $0 \leqslant \delta \leqslant 1$ and in Table 2 when $1 \leqslant \delta \leqslant 2$ for $\lambda = 1$. We see that the values of $P$ are greater than those of $P_H$ for the corresponding values of $\delta$. The values of $Q$ are greater than those of $Q_H$ in $0 \leqslant \delta < 1$ but less than those of $Q_H$ in $1 < \delta \leqslant 2$ for the corresponding values of $\delta$. All these decrease rapidly as $\delta$ increases. In Table 3 the values of $\bar{Q}$ on the interface $\delta = 1$ are given for $\lambda = 1, 2, 3$ and we observe that this difference increase with the increase in $\lambda$. In Table 4 the values of $E = -(8 \ c_2 e)/(\varrho \Omega^2 a^2)$ are given for $\lambda = 1, 2, 3$ and it is concluded that $E$ increase as $\lambda$ increases.

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## References.

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## Abstract.

*Components of stress and extension in a long rotating circular cylinder composed of non-homogeneous transversely isotropic material of uniform density with a coaxial homogeneous inner material have been obtained under the state of plane strain. Numerical values of stress-components and extension in some particular cases have been given.*

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