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Some Expansion Formulae of Bessel Functions. (**)

1. - Introduction.

In the present Note we have obtained the expansion formulae for BESSEL-functions. The orthogonality-property for the BESSEL functions and certain known results of integrals have been employed for these expansion formulae. The methods which we have followed are of great importance in the mathematical analysis.

2. - Expansion formulae.

In this section we have given the following two expansion series for BESSEL functions.

First expansion formula

$$(2.1)\quad x\frac{\sin\,[a(x+\beta)]}{(x+\beta)}\,J_{-n-(1/2)}\,(x)=\frac{\pi}{2}\sum_{r=0}^{\infty}(2r-1)\,J_{r+(1/2)}\,(\beta)\,J_{-r-(1/2)}\,(\beta)\,J_{r+(1/2)}\,(x)$$

where $n=0,1,2,\ldots,\ 2\leqslant a\leqslant\infty,\ -\infty<x<\infty$.

Second expansion formula

$$(2.2)\quad \frac{J_{\mu+(1/2)}\,(x+\alpha)}{(x+\alpha)^{\mu+(1/2)}\,x^{\nu-(1/2)}}=\sqrt{\left(\frac{\pi}{2}\right)}\,\frac{1}{\alpha^{\mu+(1/2)}}\sum_{r=0}^{\infty}\frac{(2r+1)\,(\mu+1)_r}{r!\,\alpha^r}\,J_{\mu+r+(1/2)}\,(\alpha)\;J_{r+(1/2)}\,(x)$$

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valid for $\operatorname{Re}(\mu+\nu) > -1$ and $-\infty < x < \infty$.

3. - Proofs.

This section is concerned with the proofs of the expansion formulae of (2.1) and (2.2).

(A) To prove (2.1). For $-\infty < x < \infty$, let

$$f(x) \equiv x \frac{\sin[a(x+\beta)]}{(x+\beta)} J_{-n-(1/2)}(x) = \sum_{r=0}^{\infty} A_r J_{r+(1/2)}(x) . \tag{3.1}$$

Equation (3.1) is valid since $f(x)$ is continuous and of bounded variation in the open interval $(-\infty, \infty)$. Now multiplying both sides of (3.1) by $x^{-1}J_{n+(1/2)}(x)$ and integrate with respect to x from $-\infty$ to ∞. Change the order of integration and summation on the right which is easily seen to be justified on account of the absolute convergence of the integral and summation involved during the process, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{(x+\beta)} J_{n+(1/2)}(x) J_{-n-(1/2)}(x)\, dx = \sum_{r=0}^{\infty} A_r \int_{-\infty}^{\infty} x^{-1} J_{n+(1/2)}(x) J_{r+(1/2)}(x)\, dx . \tag{3.2}$$

Now interpret the integral on the left by making use of the known result ([**1**], p. 346, (46)):

$$\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{(x+\beta)} J_{n+(1/2)}(x) J_{-n-(1/2)}(x)\, dx = \pi J_{n+(1/2)}(\beta) J_{-n-(1/2)}(\beta) ,$$

where $n = 0, 1, 2, \ldots, 2 \leqslant a < \infty$, and using the orthogonality property of BESSELL functions ([**2**], p. 391, (7)), viz.

$$\int_{-\infty}^{\infty} x^{-1} J_{n+(1/2)}(x) J_{m+(1/2)}(x)\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{2}{2n+1} & \text{if } m = n, \end{cases}$$

on the right of (3.2), we have

$$A_n = \frac{(2n+1)\pi}{2} J_{n+(1/2)}(\beta) J_{-n-(1/2)}(\beta) . \tag{3.3}$$

Expansion formula (2.1) follows immediately with the help of (3.1) and (3.3).

(*B*) Adopting the same procedure as above. For $-\infty < x < \infty$, let

$$(3.4) \qquad f(x) \equiv \frac{J_{\mu+½}(x+\alpha)}{(x+\alpha)^{\mu+½}\,x^{\alpha-½}} = \sum_{r=0}^{\infty} A_r J_{r+½}(x)\,.$$

This expansion is valid since $f(x)$ is continuous and of bounded variation in the open interval $(-\infty, \infty)$. Multiply both sides of (3.4) by $x^{-1}J_{r+(1/2)}(x)$ and integrate with respect to x. Change the order of integration and summation on the right (which we suppose to be permissible), we get

$$(3.5) \qquad \int_{-\infty}^{\infty} \frac{J^{\mu+(1/2)}(x+\alpha)}{(x+\alpha)^{\mu+½}\,x^{\nu+½}}\, J^{\nu+(1/2)}(x)\,\mathrm{d}x = \sum_{r=0}^{\infty} A_r \int_{-\infty}^{\infty} x^{-1} J_{\nu+(1/2)}(x)\, J_{r+(1/2)}(x)\,\mathrm{d}x\,.$$

Now evaluating the integral on the left by virtue of the known result ([**1**], p. 355, (32)):

$$\int_{-\infty}^{\infty} \frac{J_\mu\,[a(x+y)]}{(x+y)^\mu}\;\frac{J_\nu\,[a(x+z)]}{(x+z)^\nu}\,\mathrm{d}x = \frac{(2\pi/a)^{½}\,\Gamma(\mu+\nu)}{\Gamma(\mu+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\;\frac{J_{\mu+\nu-½}\,[a\,(y-z)]}{(y-3)^{\mu+\alpha-(1/2)}}\,,$$

where $a > 0$, Re $(\mu+\nu) > 0$, with $\mu = \mu + \frac{1}{2}$, $\nu = \nu + \frac{1}{2}$, $y = \alpha$, $a = 1$, $z = 0$ etc., and using the orthogonality property of BESSEL functions on the right, we have

$$(3.6) \qquad A_\nu = \frac{\sqrt{(\pi/2)}\,(2\nu+1)\,\Gamma(\mu+\nu+1)\,J_{\mu+r+½}(\alpha)}{\Gamma(\mu+1)\,\Gamma(\nu+1)\,\alpha^{\mu+\gamma+1/2}}\,.$$

In view of (3.4) and (3.6), we obtain the expansion formula (2.2).

References.

[1] A. Erde'lyi, *Tables of Integral Transforms*, Vol. II, McGraw-Hill, New York 1954.
[2] L. Luke Yudell, *Integrals of Bessel Functions*, McGraw-Hill, New York 1962.

Abstract

The aim of this Note is to establish certain expansion series of Bessel functions with the help of the orthogonality-property of Bessel-functions and some known integrals.

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