Riv. Mat. Univ. Parma (3) 2 (1973), 251-276

WILLIAM D. L. APPLING (*)

A Generalization of Absolute Continuity and of an Analogue of the Lebesgue Decomposition Theorem. (**)

1. - Introduction.

Suppose U is a set, $\mathbf{F}$ is a field of subsets of U, $\mathfrak{p}$ is the set of all real-valued functions defined on $\mathbf{F}$, $\mathfrak{p}_B$ is the set of all bounded elements of $\mathfrak{p}$, $\mathfrak{p}_A$ is the set of all finitely additive elements of $\mathfrak{p}_B$, $\mathfrak{p}^+$ is the set of all nonnegative-valued elements of $\mathfrak{p}$, and $\mathfrak{p}_A^+ = \mathfrak{p}_A \cap \mathfrak{p}^+$.

Suppose μ is in $\mathfrak{p}_A^+$ and A_μ is the set of all χ in $\mathfrak{p}_A$ absolutely continuous with respect to μ. In two previous papers [1], [3] the author has proved, collectively, the following theorem.

Theorem A.1. *There is a transformation* δ *from* $\mathfrak{p}_A$ *into* A_μ *such that if* χ *is in* $\mathfrak{p}_A$, *then* $\delta(\chi)$ *has the integral* (Section 2) *representation*:

$$\delta(\chi) = \int \tau(\chi)\lambda^*(\chi) ,$$

where $\tau(\chi)$ *and* $\lambda^*(\chi)$ *are elements, respectively of* $\mathfrak{p}_B$ *and* $A_\mu \cap \mathfrak{p}_A^+$, *defined by*

$$\tau(\chi)(V) = \begin{cases} 1 & \text{if } 0 \leqslant \chi(V) \\ -1 & \text{if } \chi(V) < 0 , \end{cases}$$

$$\lambda^*(\chi)(V) = \sup\{z \mid z = \int_V \min\{|\chi(J)|, K\mu(I)\}, 0 \leqslant K\} .$$

δ *is linear, and if* χ *is in* $\mathfrak{p}_A$ *and* η *is in* A_μ, *and* $\eta \neq \delta(\chi)$, *then*

$$\int_U |\chi(I) - \delta(\chi)(I)| < \int_U |\chi(I) - \eta(I)| ,$$

(*) Indirizzo. Department of Mathematics, North Texas State University, Denton, Texas 76203, U.S.A.
(**) Ricevuto: 2-IX-1970.

and

$$0 = \int_U \min\{|\chi(I) - \boldsymbol{\delta}(\chi)(I)|,\ \mu(I)\} < \int_U \min\{|\chi(I) - \eta(I)|,\ \mu(I)\}.$$

We note that Theorem A.1 is a « near point theorem » in that it says, among other things, that for each χ in $\mathfrak{p}_A$, there is an element $\boldsymbol{\delta}(\chi)$ of $\mathfrak{p}_A$, absolutely continuous with respect to μ and « closer to χ » (in the variation sense) than any other element of $\mathfrak{p}_A$ absolutely continuous to μ. Theorem A.1 further says, in the second of the above inequalities, that if χ is in $\mathfrak{p}_A$, then $\boldsymbol{\delta}(\chi)$ is the only element ι of $\mathfrak{p}_A$ absolutely continuous with respect to μ such that $\chi - \iota$ and μ are « quasi-mutually singular », a property that we shall define and characterize in Section **2**, and which the reader will easily see is equivalent, in the countably additive case, to the well-known « mutually singular » property. We therefore see that Theorem A.1 is an analogue, for the finitely additive case, of the LEBESGUE Decomposition Theorem.

In this paper we generalize Theorem A.1. We begin with the observation that A_μ is a C-set in accordance with the following definition.

Definition. The statement that M is a C-set means that $M \subseteq \mathfrak{p}_A$ and M satisfies the following two conditions:

i) If χ is in M and η is in $\mathfrak{p}_A$ and

$$\int|\chi| - \int|\eta|$$

is in $\mathfrak{p}_A^+$, then η is in M, and

ii) if β is in $\mathfrak{p}_A^+$ and α is the element of $\mathfrak{p}$ given by

$$\alpha(V) = \sup\{z \mid z = \gamma(V),\ \gamma \text{ in } M \cap \mathfrak{p}_A^+,\ \beta - \gamma \text{ in } \mathfrak{p}_A^+\},$$

then α is in $M \cap \mathfrak{p}_A^+$.

Needless to say, A_μ is just one of many examples of a C-set, and at the end of this paper (section 7) we shall list some further ones.

Suppose M is a C-set. For each χ in $\mathfrak{p}_A$, let $\boldsymbol{\tau}(\chi)$ and $\boldsymbol{\lambda}(\chi)$ be elements, respectively of $\mathfrak{p}_B$ and $M \cap \mathfrak{p}_A^+$, defined by

$$\boldsymbol{\tau}(\chi)(V) = \begin{cases} 1 & \text{if } 0 \leqslant \chi(V) \\ -1 & \text{if } \chi(V) < 0, \end{cases}$$

$$\boldsymbol{\lambda}(\chi)(V) = \sup\{z \mid z = \gamma(V),\ \gamma \text{ in } M \cap \mathfrak{p}_A^+,\ \int|\chi| - \gamma \text{ in } \mathfrak{p}_A^+\}.$$

We first prove the following generalization of the «near point» portion of Theorem A.1 (section 3):

Theorem 3.1. *There is a transformation* α *from* $\mathfrak{p}_A$ *into* M *such that if* χ *is in* $\mathfrak{p}_A$, *then* $\alpha(\chi)$ *is given by*

$$\alpha(\chi) = \int \tau(\chi)\lambda(\chi)\,.$$

α *has the property that if* η *is in* M *and* χ *is in* $\mathfrak{p}_A$ *and*

$$\eta \neq \alpha(\chi)\,,$$

then

$$\int_U |\chi(I) - \alpha(\chi)(I)| < \int_U |\chi(I) - \eta(I)|\,.$$

Then, before we pursue the question of a generalization with respect to M of the remainder of Theorem A.1, we demonstrate two further basic properties of α (section 4):

Theorem 4.1. *If each of* χ *and* η *is in* $\mathfrak{p}_A$ *and* V *is in* $\mathbf{F}$, *then*

$$\int_V |\alpha(\chi)(I) - \alpha(\eta)(I)| \leq \int_V [2\,\big||\chi(I)| - |\eta(I)|\big| + |\chi(I) - \eta(I)|]\,.$$

Theorem 4.2. *If each of* χ *and* η *is in* $\mathfrak{p}_A$, *then*

$$\int \max\{\alpha(\chi), \alpha(\eta)\} = \alpha\left(\int \max\{\chi, \eta\}\right)$$

and

$$\int \min\{\alpha(\chi), \alpha(\eta)\} = \alpha\left(\int \min\{\chi, \eta\}\right).$$

Obviously the question of the linearity of α involves the question of M's being a linear space, and indeed we show (section 6, Corollary 6.1) that if M is a linear space, then α is linear. Now it is easy to see that if α is linear, then

$$\alpha(\chi - \alpha(\chi))(U) = 0$$

for all χ in $\mathfrak{p}_A^+$. It is the question of the validity of this identity that gives us the following characterization theorem (section 5):

Theorem 5.1. *The following two statements are equivalent:*

(1) *If* χ *is in* $\mathfrak{p}_A^+$, *then*

$$\alpha(\chi - \alpha(\chi))(U) = 0\,,$$

(2) M *is a linear space.*

We then prove (section 6) the following generalization of the second inequality of Theorem A.1.

Theorem 6.1. *If* M *is a linear space and* χ *is in* $\mathfrak{p}_A$ *and* η *is in* M *and*

$$\eta \neq \alpha(\chi)\,,$$

then

$$0 = \int_U |\alpha(\chi - \alpha(\chi))(I)| < \int_U \alpha(\chi - \eta)(I)|\,.$$

2. - Preliminary theorems and definitions.

We refer the reader to sections 2, 3 and 4 of [1] and section 2 of [4] for the basic facts, notions, notations and conventions concerning subdivision, refinement, integral, $\sum$-boundedness and sum supremum and sum infimum functional. We also refer the reader to [4] for a statement of KOLMOGOROFF's [5] differential equivalence theorem and its implications about the existence and equivalence of the integrals that we shall use. When the existence of an integral or its equivalence to an integral is an easy consequence of the above mentioned material, the integral need only be written, and the proof of existence or equivalence left to the reader.

Suppose ψ is a binary operation on the real numbers and σ is a function from and into the real numbers. When convenient, we will use the convention that if each of α and β is in $\mathfrak{p}$ and each of

$$\int_U \psi(\alpha(I), \beta(I)) \qquad \text{and} \qquad \int_U \sigma(\alpha(I))$$

exists, then

$$\int \psi(\alpha, \beta) \qquad \text{and} \qquad \int \sigma(\alpha)$$

denote, respectively, the elements γ and ι of $\mathfrak{p}$ given by

$$\gamma(V) = \int_V \psi(\alpha(I), \beta(I)) \qquad \text{and} \qquad \iota(V) = \int_V \sigma(\alpha(I)) .$$

Also, when convenient, we will denote

$$\int \max\{\alpha, \beta\} \qquad \text{and} \qquad \int \min\{\alpha, \beta\}$$

respectively by

$$\alpha \vee \beta \qquad \text{and} \qquad \alpha \wedge \beta .$$

We now consider, as mentioned in the introduction, the notion of quasi-mutual singularity and give a characterization of it.

Definition. If each of η and ι is in $\mathfrak{p}_A$, than the statement that η and ι are quasi-mutually singular means that if $0 < c$; then there is an A in $\mathbf{F}$ such that

$$\max\{\int_A |\eta(I)|, \int_{U-A} |\iota(I)|\} < c .$$

Theorem 2.1. *If each of η and ι is in $\mathfrak{p}_A$, then the following two statements are equivalent*:

(1) $\int_U \min\{\int_I |\eta(J)|, \int_I |\iota(J)\} = 0$,

(2) *η and ι are quasi-mutually singular.*

Proof. Suppose 1) is true and $0 < c$. There is a subdivision $\mathfrak{D}$ of U such that

$$c > \sum_D \min\{\int_I |\eta(J)|, \int_I |\iota(J)|\} = \sum_{D_1}\int_I |\eta(J)| + \sum_{D_2}\int_I |\iota(J)| ,$$

where $\mathfrak{D}_1$ is the set (if any) of all I in $\mathfrak{D}$ such that

$$\int_I |\eta(J)| \leqslant \int_I |\iota(J)| ,$$

and $\mathfrak{D}_2 = \mathfrak{D} - \mathfrak{D}_1$. Letting $A = \cup_{D_1} I$, so that $U - A = \cup_{D_2} I$, we have that

$$c > \int_A |\eta(J)| + \int_{U-A} |\iota(J)| \geqslant \max\{\int_A |\eta(J)|, \int_{U-A} |\iota(J)|\} .$$

Therefore 1) implies 2).

Suppose 2) is true and $0 < c$. There is an A in $\mathbf{F}$ such that

$$\max\{\int_A \eta(J)|, \int_{U-A} |\iota(J)|\} < c/2 .$$

Letting $\mathfrak{D} = \{A, U-A\}$, we see that

$$\int_U \min\{\int_I |\eta(J)|, \int_I |\iota(J)|\} \leqslant \sum_D \min\{\int_I |\eta(J)|, \int_I |\iota(J)|\} \leqslant$$

$$\int_A |\eta(J)| + \int_{U-A} |\iota(J)| \leqslant c/2 + c/2 = c .$$

Therefore 2) implies 1).

Therefore 1) and 2) are equivalent, and as asserted in the introduction, this characterization gives the desired quasi-mutual singularity interpretation to the second inequality of Theorem A.1.

3. - The C-set extremal theorem.

In this section we prove Theorem 3.1, as stated in the Introduction.

Lemma 3.1. *If each of λ and η is in $M \cap \mathfrak{p}_A^+$, then*

$$\int \max\{\lambda, \eta\}$$

is in $M \cap \mathfrak{p}_A^+$.

Proof. Let β denote $\int \max\{\lambda, \eta\}$. Obviously β is in $\mathfrak{p}_A^+$.

Let α denote the element of $\mathfrak{p}$ defined by

$$\alpha(V) = \sup\{z | z = \chi(V),\ \chi \text{ in } M \cap \mathfrak{p}_A^+,\ \beta - \chi \text{ in } \mathfrak{p}_A^+\} .$$

α is in $M \cap \mathfrak{p}_A^+$, and $\beta - \alpha$ is in $\mathfrak{p}_A^+$. We therefore merely need to show that $\alpha - \beta$ is in $\mathfrak{p}_A^+$ in order to prove our lemma. Obviously each of $\beta - \lambda$ and $\beta - \eta$ is in $\mathfrak{p}_A^+$, so that each of $\alpha - \lambda$ and $\alpha - \eta$ is in $\mathfrak{p}_A^+$, which immediately implies that $\alpha - \int \max\{\lambda, \eta\}$ is in $\mathfrak{p}_A^+$.

Therefore $\alpha = \beta$, so that β is in M and therefore in $M \cap \mathfrak{p}_A^+$.

Lemma 3.2. *If ι is in $\mathfrak{p}_A^+$ and η is in $M \cap \mathfrak{p}_A^+$ and*

$$\eta \neq \boldsymbol{\lambda}(\iota) ,$$

then

$$\int_U |\iota(I) - \lambda(\iota)(I)| < \int_U |\iota(I) - \eta(I)| .$$

Proof. Let λ and β denote, respectively, $\boldsymbol{\lambda}(\iota)$ and $\int \min\{\iota, \eta\}$.

From i) of the definition of a C-set it follows that β is in M, and we see that β is in $\mathfrak{p}_A^+$. Furthermore, we see that $\lambda - \beta$ is in $\mathfrak{p}_A^+$. Therefore, for each V in $\mathbf{F}$,

$$\int_V |\iota(I) - \eta(I)| = \int_V \max\{\iota(I), \eta(I)\} - \int_V \min\{\iota(I), \eta(I)\} \geqslant$$

$$\geqslant \iota(V) - \lambda(V) .$$

If $\lambda - \eta$ is not in $\mathfrak{p}_A^+$, then $\iota - \eta$ is not in $\mathfrak{p}_A^+$, so that for some V in $\mathbf{F}$,

$$\eta(V) > \iota(V) ,$$

so that

$$\int_V \max\{\iota(I), \eta(I)\} - \int_V \min\{\iota(I), \eta(I)\} \geqslant \eta(V) - \lambda(V) > \iota(V) - \lambda(V) ,$$

so that

$$\int_U |\iota(I) - \eta(I)| = \int_{U-V} |\iota(I) - \eta(I)| + \int_V |\iota(I) - \eta(I)| >$$

$$> \iota(U - V) - \lambda(U - V) + \iota(V) - \lambda(V) = \iota(U) - \lambda(U) .$$

If $\lambda - \eta$ is in $\mathfrak{p}_A^+$, then for some V in $\mathbf{F}$,

$$\eta(V) < \lambda(V) ,$$

so that

$$\int_V |\iota(I) - \eta(I)| \geqslant \iota(V) - \eta(V) > \iota(V) - \lambda(V) ,$$

so that

$$\int_U |\iota(I) - \eta(I)| = \int_{U-V} |\iota(I) - \eta(I)| + \int_V |\iota(I) - \eta(I)| >$$

$$> \iota(U - V) - \lambda(U - V) + \iota(V) - \lambda(V) = \iota(U) - \lambda(U) .$$

Therefore

$$\int_U |\iota(I) - \eta(I)| > \iota(U) - \lambda(U) = \int_U |\iota(I) - \lambda(\iota)(I)| ,$$

since $\iota - \lambda(\iota)$ is in $\mathfrak{p}_A^+$.

We now prove Theorem 3.1.

Proof of Theorem 3.1. Suppose χ is in $\mathfrak{p}_A$. We see that $\int|\chi|$ is in $\mathfrak{p}_A^+$, $\boldsymbol{\lambda}(\chi)$ is in $M \cap \mathfrak{p}_A^+$, and that $\boldsymbol{\tau}(\chi)$ is in $\mathfrak{p}_B$.

We see that for each X in $\mathbf{F}$,

$$\boldsymbol{\tau}(\chi)(X)\,\chi(X) = |\chi(X)|\,,$$

which implies that for each V in $\mathbf{F}$,

$$\int_V \boldsymbol{\tau}(\chi)(I)\,\chi(I)$$

exists and is

$$\int_V |\chi(I)|\,.$$

Since

$$\boldsymbol{\tau}(\chi)(V)\,\boldsymbol{\tau}(\chi)(V) = 1$$

for each V in $\mathbf{F}$, it follows that

$$\int_U \boldsymbol{\tau}(\chi)(I)\,\boldsymbol{\tau}(\chi)(I)\,\chi(I)$$

exists, so that

$$\int_U \boldsymbol{\tau}(\chi)(I) \int_I \boldsymbol{\tau}(\chi)(J)\,\chi(J)$$

exists. Since $\int|\chi| - \lambda(\chi)$ is in $\mathfrak{p}_A^+$, it follows that

$$\int_U \boldsymbol{\tau}(\chi)(I)\,\boldsymbol{\lambda}(\chi)(I)$$

exists.

We therefore see that there is a transformation $\boldsymbol{\alpha}$ from $\mathfrak{p}_A$ into $\mathfrak{p}_A$ such that if χ is in $\mathfrak{p}_A$, then $\boldsymbol{\alpha}(\chi)$ is given by

$$\boldsymbol{\alpha}(\chi) = \int \boldsymbol{\tau}(\chi)\,\boldsymbol{\lambda}(\chi)\,.$$

We immediately see that if χ is in $\mathfrak{p}_A$, then $\boldsymbol{\alpha}(\chi)$ is in M from i) of the definition of a C-set, and we also see that

$$\int |\boldsymbol{\alpha}(\chi)| = \boldsymbol{\lambda}(\chi) = \boldsymbol{\alpha}(\int |\chi|)\,,$$

as well as the fact that if ι is in M, then

$$\boldsymbol{\alpha}(\iota) = \iota\,.$$

Now suppose χ is in $\mathfrak{p}_A$, η is in M and

$$\eta \neq \boldsymbol{\alpha}(\chi)\,.$$

Let τ and λ denote, respectively, $\boldsymbol{\tau}(\chi)$ and $\boldsymbol{\lambda}(\chi)$.
$\int|\eta|$ is in $M \cap \mathfrak{p}_A^+$, and it is either true that

$$\int |\eta| = \lambda\,,$$

or not.

Suppose that

$$\int |\eta| \neq \lambda\,.$$

Then

$$\int_U |\chi(I) - \eta(I)| \geqslant \int_U \big||\chi(I)| - |\eta(I)|\big| =$$

$$= \int_U \Big| \int_I |\chi(J)| - \int_I |\eta(J)| \Big| > \int_U \Big| [\int_I |\chi(J)|] - \lambda(I) \Big|$$

by Lemma 3.2. Now

$$\int_U \Big| [\int_I |\chi(J)|] - \lambda(I) \Big| = \int_U \big||\chi(I)| - \lambda(I)\big| =$$

$$= \int_U |\tau(I)|\,|\tau(I)\,\chi(I) - \lambda(I)| = \int_U |\tau(I)\,\tau(I)\,\chi(I) - \tau(I)\,\lambda(I)| =$$

$$= \int_U |\chi(I) - \int_I \tau(J)\,\lambda(J)| = \int_U |\chi(I) - \boldsymbol{\alpha}(\chi)(I)|\,,$$

so that

$$\int_U |\chi(I) - \eta(I)| > \int_U |\chi(I) - \boldsymbol{\alpha}(\chi)(I)|\,.$$

Now suppose that

$$\int |\eta| = \lambda .$$

We see that for each V in $\mathbf{F}$,

$$|\eta(V)| \leqslant \lambda(V) ,$$

which implies that

$$\int \tau(I)\eta(I)$$

exists.

Now

$$\int_U |\chi(I) - \eta(I)| = \int_U |\tau(I)|\,|\chi(I) - \eta(I)| =$$

$$= \int_U |\tau(I)\chi(I) - \tau(I)\eta(I)| = \int_U \Big|\big[\int_I \chi(J)|\big] - \lambda(I) + \lambda(I) - \tau(I)\eta(I)\Big| =$$

$$= \int_U \Big|\big[\int_I |\chi(J)|\big] - \lambda(I)\Big| + \int_U |\lambda(I) - \tau(I)\eta(I)| =$$

$$= \int_U |\tau(I)|\,|\tau(I)\chi(I) - \lambda(I)| + \int_U |\tau(I)|\,|\lambda(I) - \tau(I)\eta(I)| =$$

$$= \int_U |\tau(I)^2\chi(I) - \tau(I)\lambda(I)| + \int_U |\tau(I)\lambda(I) - \tau(I)^2\eta(I)| =$$

$$= \int_U |\chi(I) - \int_I \tau(J)\lambda(J)| + \int_U \Big|\big[\int_I \tau(J)\lambda(J)|\big] - \eta(I)\Big| =$$

$$= \int_U |\chi(I) - \boldsymbol{\alpha}(\chi)(I)| + \int_U |\boldsymbol{\alpha}(\chi)(I) - \eta(I)| > \int_U |\chi(I) - \boldsymbol{\alpha}(\chi)(I)| .$$

Therefore

$$\int_U |\chi(I) - \eta(I)| > \int_U |\chi(I) - \boldsymbol{\alpha}(\chi)(I)| .$$

4. - A continuity and a functional equation theorem.

We precede the proof of Theorem 4.1 by four lemmas.

Lemma 4.1. *If each of χ and η is in $\mathfrak{p}_A^+$, then*

$$\int \max\{\boldsymbol{\lambda}(\chi), \boldsymbol{\lambda}(\eta)\} = \boldsymbol{\lambda}\big(\int \max\{\chi, \eta\}\big) \quad \text{and} \quad \int \min\{\boldsymbol{\lambda}(\chi), \boldsymbol{\lambda}(\eta)\} = \boldsymbol{\lambda}\big(\int \min\{\chi, \eta\}\big).$$

Proof. Suppose $0 < c$ and V is in $\mathbf{F}$. There is an ι and a ψ, each in $M \cap \mathfrak{p}_A^+$ such that each of $\chi - \iota$ and $\eta - \psi$ is in $\mathfrak{p}_A^+$ and

$$\boldsymbol{\lambda}(\chi)(V) - \iota(V) < c/2 \qquad \text{and} \qquad \boldsymbol{\lambda}(\eta)(V) - \psi(V) < c/2 \, .$$

Each of $\boldsymbol{\lambda}(\chi) - \iota$ and $\boldsymbol{\lambda}(\eta) - \psi$ is in $\mathfrak{p}_A^+$, $\int\max\{\iota, \psi\}$ is in $M \cap \mathfrak{p}_A^+$, and $\int\max\{\chi, \eta\} - \int\max\{\iota, \psi\}$ is in $\mathfrak{p}_A^+$, so that, respectively each of $\int\max\{\boldsymbol{\lambda}(\chi), \boldsymbol{\lambda}(\eta)\} - \int\max\{\iota, \psi\}$ and $\boldsymbol{\lambda}(\int\max\{\chi, \eta\}) - \int\max\{\iota, \psi\}$ is in $\mathfrak{p}_A^+$. Furthermore,

$$\int_V \max\{\boldsymbol{\lambda}(\chi)(I), \boldsymbol{\lambda}(\eta)(I)\} - \int_V \max\{\iota(I), \psi(I)\} < c/2 + c/2 = c \, ,$$

so that

$$\int_V \max\{\boldsymbol{\lambda}(\chi)(I), \boldsymbol{\lambda}(\eta)(I)\} < c + \int_V \max\{\iota(I), \psi(I)\} \leqslant$$

$$\leqslant c + \boldsymbol{\lambda}(\int\max\{\chi, \eta\})(V) \, .$$

Therefore $\boldsymbol{\lambda}(\int\max\{\chi, \eta\}) - \int\max\{\boldsymbol{\lambda}(\chi), \boldsymbol{\lambda}(\eta)\}$ is in $\mathfrak{p}_A^+$.

Now, suppose for some V in $\mathbf{F}$,

$$\int_V \max\{\boldsymbol{\lambda}(\chi)(I), \boldsymbol{\lambda}(\eta)(I)\} < \boldsymbol{\lambda}\left(\int\max\{\chi, \eta\}\right)(V) \, .$$

There is an σ in $M \cap \mathfrak{p}_A^+$ such that $\int\max\{\chi, \eta\} - \sigma$ is in $\mathfrak{p}_A^+$ and

$$\int_V \max\{\boldsymbol{\lambda}(\chi)(I), \boldsymbol{\lambda}(\eta)(I)\} < \sigma(V) \, .$$

Each of $\int\min\{\chi, \sigma\}$ and $\int\min\{\eta, \sigma\}$ is in $M \cap \mathfrak{p}_A^+$, and each of $\chi - \int\min\{\chi, \sigma\}$ and $\eta - \int\min\{\eta, \sigma\}$ is in $\mathfrak{p}_A^+$, so that each of $\boldsymbol{\lambda}(\chi) - \int\min\{\chi, \sigma\}$ and $\boldsymbol{\lambda}(\eta) - \int\min\{\eta, \sigma\}$ is in $\mathfrak{p}_A^+$, which implies that $\mathfrak{p}_A^+$ contains $\int\max\{\boldsymbol{\lambda}(\chi), \boldsymbol{\lambda}(\eta)\} - \int\max\{\int\min\{\chi, \sigma\}, \int\min\{\eta, \sigma\}\}$. Now $\int\max\{\int\min\{\chi, \sigma\}, \int\min\{\eta, \sigma\}\} = \int\min\{\int\max\{\chi, \eta\}, \sigma\} = s$, so that $\int\max\{\boldsymbol{\lambda}(\chi), \boldsymbol{\lambda}(\eta)\} - \sigma$ is in $\mathfrak{p}_A^+$, a contradiction. Therefore $\int\max\{\boldsymbol{\lambda}(\chi), \boldsymbol{\lambda}(\eta)\} - \boldsymbol{\lambda}(\int\max\{\chi, \eta\})$ is in $\mathfrak{p}_A^+$.

Therefore

$$\int\max\{\boldsymbol{\lambda}(\chi), \boldsymbol{\lambda}(\eta)\} = \boldsymbol{\lambda}\left(\int\max\{\chi, \eta\}\right) .$$

To show that

$$\int\min\{\boldsymbol{\lambda}(\chi), \boldsymbol{\lambda}(\eta)\} = \boldsymbol{\lambda}\left(\int\min\{\chi, \eta\}\right) ,$$

we need only modify the above argument by changing «max» to «min».

Lemma 4.2. *If each of χ, η and $\chi - \eta$ is in $\mathfrak{p}_A^+$, then so is $\chi - \eta - [\boldsymbol{\lambda}(\chi) - \boldsymbol{\lambda}(\eta)]$.*

Proof. Obviously each of $\boldsymbol{\lambda}(\chi) - \boldsymbol{\lambda}(\eta)$ and $\eta - \boldsymbol{\lambda}(\eta)$ is in $\mathfrak{p}_A^+$ and therefore so is $\int \min\{\boldsymbol{\lambda}(\chi), \eta\} - \boldsymbol{\lambda}(\eta)$. Now, $\eta - \int \min\{\boldsymbol{\lambda}(\chi), \eta\}$ is in $\mathfrak{p}_A^+$ and $\int \min\{\boldsymbol{\lambda}(\chi), \eta\}$ is obviously in $M \cap \mathfrak{p}_A^+$, so that $\boldsymbol{\lambda}(\eta) - \int \min\{\boldsymbol{\lambda}(\chi), \eta\}$ is in $\mathfrak{p}_A^+$. Therefore

$$\boldsymbol{\lambda}(\eta) = \int \min\{\boldsymbol{\lambda}(\chi), \eta\}\,.$$

Hence, if V is in $\mathbf{F}$, then

$$\boldsymbol{\lambda}(\chi)(V) - \boldsymbol{\lambda}(\eta)(V) = \int_V \min\{\boldsymbol{\lambda}(\chi)(I), \chi(I)\} - \int_V \min\{\boldsymbol{\lambda}(\chi)(I), \eta(I)\}$$

$$\leqslant \chi(V) - \eta(V)\,.$$

Therefore $\chi - \eta - [\boldsymbol{\lambda}(\chi) - \boldsymbol{\lambda}(\eta)]$ is in $\mathfrak{p}_A^+$.

Lemma 4.3. *If each of ι and ψ is in $\mathfrak{p}_A^+$, then so is*

$$\int|\iota - \psi| - \int|\boldsymbol{\lambda}(\iota) - \boldsymbol{\lambda}(\psi)|\,.$$

Proof:

$$\int|\iota - \psi| - \int|\boldsymbol{\lambda}(\iota) - \boldsymbol{\lambda}(\psi)| = \int \max\{\iota, \psi\} - \int \min\{\iota, \psi\} -$$

$$[\int \max\{\boldsymbol{\lambda}(\iota), \boldsymbol{\lambda}(\psi)\} - \int \min\{\boldsymbol{\lambda}(\iota), \boldsymbol{\lambda}(\psi)\}]\,,$$

which, by Lemma 4.1, is

$$\int \max\{\iota, \psi\} - \int \min\{\iota, \psi\} - [\boldsymbol{\lambda}(\int \max\{\iota, \psi\}) - \boldsymbol{\lambda}(\int \min\{\iota, \psi\})]\,,$$

which, by Lemma 4.2, is in $\mathfrak{p}_A^+$.

Lemma 4.4. *If each of χ and η is in $\mathfrak{p}_A$ and V is in $\mathbf{F}$, then*

$$\int_V \mathbf{s}^*(|\boldsymbol{\tau}(\chi) - \boldsymbol{\tau}(\eta)| \int|\chi|)(I) \leqslant \int_V [\,\big||\eta(I)| - |\chi(I)|\big| + |\eta(I) - \chi(I)|\,]\,.$$

Proof. Suppose I is in $\mathbf{F}$. If $\mathfrak{D}$ is a subdivision of I, then

$$\sum_D \Big|\boldsymbol{\tau}(\eta)(J)\int_J|\eta| - \boldsymbol{\tau}(\chi)(J)\int_J|\chi|\Big| =$$

$$= \sum_D \Big|\{\int_J|\eta| - \int_J|\chi|\}\boldsymbol{\tau}(\eta)(J) + \{\boldsymbol{\tau}(\eta)(J) - \boldsymbol{\tau}(\chi)(J)\}\int_J|\chi|\Big| \geqslant$$

$$\geqslant \sum_{D} |\tau(\eta)(J) - \tau(\chi)(J)| \int_{J} |\chi| - \sum_{D} \left| \int_{J} |\eta| - \int_{J} |\chi| \right| |\tau(\eta)(J)| .$$

This implies that

$$s^* \left(|\tau(\eta) - \tau(\chi)| \int |\chi| \right)(I) \leqslant s^* \left(\left| \int |\eta| - \int |\chi| \right| |\tau(\eta)| (I) + \right.$$
$$+ s^* \left(\left| \tau(\eta) \int |\eta| - \tau(\chi) \int |\chi| \right| \right)(I) .$$

Therefore

$$\int_{V} s^* \left(|\tau(\eta) - \tau(\chi)| \int |\chi| \right)(I) \leqslant \int_{V} s^* \left(\left| \int |\eta| - \int |\chi| \right| |\tau(\eta)| \right)(I) +$$
$$\int_{V} + s^* \left(\left| \tau(\eta) \int |\eta| - \tau(\chi) \int |\chi| \right| \right)(I) = \int_{V} \left[\big| |\eta(I)| - |\chi(I)| \big| + |\eta(I) - \chi(I)| \right] .$$

We now prove Theorem 4.1, as stated in the introduction.

Proof of Theorem 4.1. Suppose I is in $\mathbf{F}$. If $\mathfrak{D}$ is a subdivision of I, then

$$\sum_{D} |\tau(\eta)(J)\lambda(\eta)(J) - \tau(\chi)(J)\lambda(\chi)(J)| \leqslant$$
$$\leqslant \sum_{D} |\tau(\eta)(J)| |\lambda(\eta)(J) - \lambda(\chi)(J)| + \sum_{D} |\tau(\eta)(J) - \tau(\chi)(J)| \lambda(\chi)(J) .$$

This implies that

$$s^*(|\tau(\eta)\lambda(\eta) - \tau(\chi)\lambda(\chi)|)(I) \leqslant s^*(|\lambda(\eta) - \lambda(\chi)|)(I) +$$
$$+ s^*(|\tau(\eta) - \tau(\chi)|\lambda(\chi))(I) .$$

Therefore, if V is in $\mathbf{F}$, then

$$\int_{V} |\alpha(\eta)(I) - \alpha(\chi)(I)| = \int_{V} |\tau(\eta)(I)\lambda(\eta)(I) - \tau(\chi)(I)\lambda(\chi)(I)| =$$
$$= \int_{V} s^*(|\tau(\eta)\lambda(\eta) - \tau(\chi)\lambda(\chi)|)(I) \leqslant \int_{V} s^*(|\lambda(\eta) - \lambda(\chi)|)(I) +$$
$$+ \int_{V} s^*(|\tau(\eta) - \tau(\chi)|\lambda(\chi))(I) \leqslant \int_{V} s^*(|\lambda(\eta) - \lambda(\chi)|)(I) +$$
$$+ \int_{V} s^*\left(|\tau(\eta) - \tau(\chi)| \int |\chi|\right)(I) = \int_{V} \left|\lambda\left(\int |\eta|\right)(I) - \lambda\left(\int |\chi|\right)(I)\right| +$$

$$+\int_V s^*(|\tau(\eta)-\tau(\chi)|\int|\chi|)(I) \leqslant \int_V \Big|\int_I|\eta|-\int_I|\chi|\Big| +$$

$$+\int_V [\, \big| |\eta(I)|-|\chi(I)| \big| + |\eta(I)-\chi(I)|\,] =$$

$$=\int_V [2\, \big| |\eta(I)|-|\chi(I)| \big| + |\eta(I)-\chi(I)|] \, .$$

Therefore

$$\int_V |\alpha(\chi)(I)-\alpha(\eta)(I)| \leqslant \int_V [2\, \big||\chi(I)|-|\eta(I)|\big| + |\chi(I)-\eta(I)|] \, .$$

We now state and prove three lemmas that we will use to prove Theorem 4.2.

Lemma 4.5. *If μ is in $\mathfrak{p}_A^+$, $a<b$ and β is a function from* **F** *into $\{a, b\}$ such that $\int_U \beta(I)\mu(I)$ exists, then*

$$\int_U\int_V |\beta(V)\mu(I)-\int_I \beta(J)\mu(J)| = 0 \, .$$

Proof. Suppose $0<c$. By the theorem of KOLMOGOROFF [**5**] previously mentioned, there is a subdivision $\mathfrak{D}$ of U such that if $\mathfrak{E}$ is a refinement of $\mathfrak{D}$ then

$$\sum_E |\beta(V)\mu(V)-\int_V \beta(I)\mu(I)| < c \, .$$

Now suppose $\mathfrak{E}$ is a refinement of $\mathfrak{D}$. Suppose V is in $\mathfrak{E}$. From the hypothesis it follows that either $\beta(V)\leqslant\beta(J')$ for all J' in **F**, or $\beta(J')\leqslant\beta(V)$ for all S' in **F**. In the first case

$$\int_V |\beta(V)\mu(I)-\int_I \beta(J)\mu(J)| = \int_V [\int_I \beta(J)\mu(J)-\beta(V)\mu(I)] =$$

$$=[\int_V \beta(J)\mu(J)]-\beta(V)\mu(V) = |\beta(V)\mu(V)-\int_V \beta(J)\mu(J)| \, ;$$

in the second case

$$\int_V |\beta(V)\mu(I)-\int_I \beta(J)\mu(J)| = \int_V [\beta(V)\mu(I)-\int_I \beta(J)\mu(J)] =$$

$$=\beta(V)\mu(V)-\int_V \beta(J)\mu(J) = |\beta(V)\mu(V)-\int_V \beta(J)\mu(J)| \, .$$

Therefore

$$\sum_E \int_V |\beta(V)\mu(I) - \int_I \beta(J)\mu(J)| = \sum_E |\beta(V)\mu(V) - \int_V \beta(J)\mu(J)| < c .$$

Therefore

$$\int_U \int_V |\beta(V)\mu(I) - \int_I \beta(J)\mu(J)| = 0 .$$

Lemma 4.6. *If* χ *is in* $\mathfrak{p}_A^+$, β *is in* $\mathfrak{p}$, $\beta[1-\beta] = 0$, *and* $\int_U \beta(I)\chi(I)$ *exists, then*

$$\boldsymbol{\lambda}(\int \beta\chi) = \int \beta\lambda(\chi) .$$

Proof. We first observe that if d is 0 or 1, then $d\boldsymbol{\lambda}(\chi) = \boldsymbol{\lambda}(d\chi)$. We also easily see that

$$\int \beta(I)\boldsymbol{\lambda}(\chi)(I)$$

exists and that $\int\beta\boldsymbol{\lambda}(\chi)$ is in $M \cap \mathfrak{p}_A^+$.

Now suppose $0 < c$. From Lemma 4.5 it follows that there is a subdivision $\mathfrak{D}$ of U such that

$$\sum_D \int_V |\beta(V)\chi(I) - \int_I \beta(J)\chi(J)| < c/3$$

and

$$\sum_U \int_V | \beta(V)\boldsymbol{\lambda}(\chi)(I) - \int_I \beta(J)\boldsymbol{\lambda}(\chi)(J)| < c/3 ,$$

so that

$$\int_U |\boldsymbol{\lambda}(\int\beta\chi)(I) - \int_I \beta(J)\boldsymbol{\lambda}(\chi)(J)| \leqslant \sum_D \int_V |\boldsymbol{\lambda}(\int\beta\chi)(I) - \boldsymbol{\lambda}(\beta(V)\chi)(I)| +$$

$$+ \sum_D \int_V |\beta(V)\boldsymbol{\lambda}(\chi)(I) - \int_I \beta(J)\boldsymbol{\lambda}(\chi)(J)| \leqslant$$

$$\leqslant \sum_D \int_V | \int_I \beta(J)\chi(J) - \beta(V)\chi(I)| + c/3 \leqslant 2c/3 < c .$$

Therefore

$$\int_U |\boldsymbol{\lambda}(\beta\chi)(I) - \int_I \beta(J)\boldsymbol{\lambda}(\chi)(J)| = 0 ,$$

so that

$$\boldsymbol{\lambda}(\int \beta\chi) = \int \beta\boldsymbol{\lambda}(\chi) .$$

Lemma 4.7. *If η is in $\mathfrak{p}_A$, then*

$$\boldsymbol{\alpha}(\int \max\{\eta, 0\}) = \int \max\{\boldsymbol{\alpha}(\eta), 0\}$$

and

$$\boldsymbol{\alpha}(\int \min\{\eta, 0\}) = \int \min\{\boldsymbol{\alpha}(\eta), 0\} .$$

Proof. If χ is in $\mathfrak{p}_A$, then

$$\boldsymbol{\alpha}(\int \max\{\chi, 0\}) = \boldsymbol{\lambda}(\int \max\{\boldsymbol{\tau}(\chi)|\chi|, 0\}) = \lambda(\int \max\{[(\boldsymbol{\tau}(\chi)+1)/2]|\chi|, 0\},$$

which by Lemma 4.1 is

$$\int \max\{\boldsymbol{\lambda}(\int [(\boldsymbol{\tau}(\chi)+1)/2] \int |\chi|), 0\} .$$

Obviously for each I in $\mathbf{F}$, $(\boldsymbol{\tau}(\chi)(I)+1)/2$ is 0 or 1. Therefore by Lemma 4.6,

$$\int \max\{\boldsymbol{\lambda}(\int\{[(\boldsymbol{\tau}(\chi)+1)/2] \int |\chi|\}), 0\} = \int \max\{\int [(\boldsymbol{\tau}(\chi)+1)/2]\boldsymbol{\lambda}(\chi), 0\} =$$
$$= \int \max\{\boldsymbol{\tau}(\chi)\boldsymbol{\lambda}(\chi), 0\} = \int \max\{\boldsymbol{\alpha}(\chi), 0\} .$$

Before proceeding we note the fact, and leave the easy proof to the reader, that if χ is in $\mathfrak{p}_A$, then

$$\boldsymbol{\alpha}(-\chi) = -\boldsymbol{\alpha}(\chi) .$$

From the above two paragraphs we see that if ι is in $\mathfrak{p}_A$, then

$$\boldsymbol{\alpha}(\int \min\{\iota, 0\}) = \boldsymbol{\alpha}(-\int \max\{-\iota, 0\}) = -\boldsymbol{\alpha}(\int \max\{-\iota, 0\}) =$$
$$-\int \max\{\boldsymbol{\alpha}(-\iota), 0\} = \int \min\{-\boldsymbol{\alpha}(-\iota), 0\} = \int \min\{\boldsymbol{\alpha}(\iota), 0\} .$$

We now prove Theorem 4.2, as stated in the introduction.

Proof of Theorem 4.2. If each of ι and ψ is in $\mathfrak{p}_A$, then

$$\boldsymbol{\alpha}(\iota \vee \psi) \vee 0 = \boldsymbol{\alpha}([\iota \vee \psi] \vee 0) = \boldsymbol{\alpha}([\iota \vee 0] \vee [\psi \vee 0]) = \boldsymbol{\lambda}([\iota \vee 0] \vee [\psi \vee 0]) =$$
$$\boldsymbol{\lambda}(\iota \vee 0) \vee \boldsymbol{\lambda}(\psi \vee 0) = \boldsymbol{\alpha}(\iota \vee 0) \vee \boldsymbol{\alpha}(\psi \vee 0) = [\boldsymbol{\alpha}(\iota) \vee 0] \vee [\boldsymbol{\alpha}(\psi) \vee 0] =$$
$$[\boldsymbol{\alpha}(\iota) \vee \boldsymbol{\alpha}(\psi)] \vee 0 ,$$

and

$$\alpha(\iota\vee\psi)\wedge 0 = \alpha[(\iota\vee\psi)\wedge 0] = \alpha[(\iota\wedge 0)\vee(\psi\wedge 0)] =$$

$$\alpha[-\{(-[\iota\wedge 0])\wedge(-[\psi\wedge 0])\}] = -\alpha[(-[\iota\wedge 0])\wedge(-[\psi\wedge 0])] =$$

$$-\lambda[(-[\iota\wedge 0])\wedge(-[\psi\wedge 0])] = -\{\lambda(-[\iota\wedge 0])\wedge\lambda(-[\psi\wedge 0])\} =$$

$$-\{\alpha(-[\iota\wedge 0])\wedge\alpha(-[\psi\wedge 0])\} = -\{(-\alpha(\iota\wedge 0))\wedge(-\alpha(\psi\wedge 0))\} =$$

$$\alpha(\iota\wedge 0)\vee\alpha(\psi\wedge 0) = [\alpha(\iota)\wedge 0]\vee[\alpha(\psi)\wedge 0] = [\alpha(\iota)\vee\alpha(\psi)]\wedge 0\ ,$$

so that from the identity

$$\chi = \max\{\chi, 0\} + \min\{\chi, 0\}\ ,$$

we have

$$\alpha(\iota\vee\psi) = \alpha(\iota)\vee\alpha(\psi)\ .$$

From the above paragraph we have that if each of σ and ν is in $\mathfrak{p}_A$, then

$$\alpha(\sigma\wedge\nu) = \alpha(-[(-\sigma)\vee(-\nu)]) = -\alpha((-\sigma)\vee(-\nu)) = -[\alpha(-\sigma)\vee\alpha(-\nu)] =$$

$$-[(-\alpha(\sigma))\vee(-\alpha(\nu))] = \alpha(\sigma)\wedge\alpha(\nu)\ .$$

5. - The C-set-linear space characterization theorem.

In this section we prove Theorem 5.1, as stated in the Introduction.

Proof of Theorem 5.1. First, suppose 1) is true and γ is in $M\cap\mathfrak{p}_A^+$. Obviously each of

$$2\gamma-\alpha(2\gamma) \qquad \text{and} \qquad \alpha(2\gamma)-\gamma$$

is in $\mathfrak{p}_A^+$, which implies that

$$\gamma-[2\gamma-\alpha(2\gamma)]$$

is in $\mathfrak{p}_A^+$, so that

$$2\gamma-\alpha(2\gamma)$$

is in $M \cap \mathfrak{p}_A^+$. Therefore

$$[2\gamma - \alpha(2\gamma)](U) = \alpha(2\gamma - \alpha(2\gamma))(U) = 0\ ,$$

so that

$$2\gamma = \alpha(2\gamma)\ ,$$

so that 2γ is in $M \cap \mathfrak{p}_B^+$.

Now suppose that each of χ and η is in M and c is a number. Let χ^* and η^* denote, respectively, $\int|\chi|$ and $\int|\eta|$. By Lemma 3.1, the function γ^* given by

$$\gamma^* = \int \max\{\chi^*, \eta^*\}$$

is in $M \cap \mathfrak{p}_A^+$, so that by the preceding paragraph, $2\gamma^*$ is in $M \cap \mathfrak{p}_A^+$. Since

$$2\gamma^* - [\chi^* + \eta^*]$$

is in $\mathfrak{p}_A^+$, it follows that

$$\chi^* + \eta^*$$

is in $M \cap \mathfrak{p}_A^+$, and therefore, since

$$\chi^* + \eta^* - \int|\chi + \eta|$$

is in $\mathfrak{p}_A^+$, it follows that $\chi+\eta$ is in M. Now there is a positive integer n such that

$$|c| < 2^n.$$

By induction,

$$2^n \eta^*$$

is in $M \cap \mathfrak{p}_A^+$. Furthermore,

$$2^n \eta^* - \int|c\eta|$$

is in $\mathfrak{p}_A$, so that $c\eta$ is therefore in M. Therefore M is a linear space.

Therefore 1) implies 2).

Now suppose 2) is true and χ is in $\mathfrak{p}_A$. Obviously each of

$$\chi - \alpha(\chi) \qquad \text{and} \qquad [\chi - \alpha(\chi)] - \alpha[\chi - \alpha(\chi)]$$

is in $\mathfrak{p}_A^+$, so that

$$\chi - \{\alpha(\chi) + \alpha[\chi - \alpha(\chi)]\}$$

is in $\mathfrak{p}_A^+$. Since

$$\alpha(\chi) + \alpha[\chi - \alpha(\chi)]$$

is in $M \cap \mathfrak{p}_A^+$, it follows that $\mathfrak{p}_A^+$ contains

$$\alpha(\chi) - \{\alpha(\chi) + \alpha[\chi - \alpha(\chi)]\}\,,$$

which is

$$-\,\alpha[\chi - \alpha(\chi)]\,.$$

Since $\alpha[\chi - \alpha(\chi)]$ is in $\mathfrak{p}_A^+$, it follows that

$$\alpha[\chi - \alpha(\chi)](U) = 0\,.$$

Therefore 2) implies 1).
Therefore 1) and 2) are equivalent.

6. - Two linear space theorems.

We suppose throughout this section that the C-set M is a linear space.
In this section we prove Theorem 6.1 and Corollary 6.1, as stated in the introduction.
We begin with a lemma.

Lemma 6.1. *If each of χ and η is in $\mathfrak{p}_A^+$ and $0 \leqslant c$, then each of*

$$\alpha(\chi) + \alpha(\eta) - \alpha(\chi + \eta) \qquad \textit{and} \qquad c\alpha(\chi) - \alpha(c\chi)$$

is in $\mathfrak{p}_A^+$.

Proof. We first show that

$$\alpha(\chi) + \alpha(\eta) - \alpha(\chi + \eta)$$

is in $\mathfrak{p}_A^+$.

Suppose, on the contrary, that for some X in $\mathbf{F}$,

$$\alpha(\chi)(X) + \alpha(\eta)(X) < \alpha(\chi + \eta)(X) .$$

There is a γ in $M \cap \mathfrak{p}_A^+$ such that $\chi + \eta - \gamma$ is in $\mathfrak{p}_A^+$ and such that

$$\alpha(\chi)(X) + \alpha(\eta)(X) < \gamma(X) .$$

Let ϱ and ν denote, respectively,

$$\int \min\{\chi, \gamma\} \qquad \text{and} \qquad \int \min\{\eta, \gamma\} .$$

Obviously each of ϱ and ν is in $M \cap \mathfrak{p}_A^+$, so that $\varrho + \nu$ is in $M \cap \mathfrak{p}_A^+$. Each of $\chi - \varrho$ and $\eta - \nu$ is in $\mathfrak{p}_A^+$, so that each of $\alpha(\chi) - \varrho$ and $\alpha(\eta) - \nu$ is in $\mathfrak{p}_A^+$. Since $\chi + \eta - \gamma$ is in $\mathfrak{p}_A^+$, it follows that if V is in $\mathbf{F}$, then

$$\varrho(V) + \nu(V) = \int_V [\min\{\chi(I), \gamma(I)\} + \min\{\eta(I), \gamma(I)\}] \geqslant$$

$$\int_V \gamma(I) = \gamma(V) .$$

Therefore, since $\mathfrak{p}_A^+$ contains each of $\varrho + \nu - \gamma$ and $\alpha(\chi) - \varrho + \alpha(\eta) - \nu$, which is $\alpha(\chi) + \alpha(\eta) - [\varrho + \nu]$, it follows that $\alpha(\chi) + \alpha(\eta) - \gamma$ is in $\mathfrak{p}_A^+$, so that

$$\gamma(X) \leqslant \alpha(\chi)(X) + \alpha(\eta)(X) ,$$

a contradiction. Therefore $\alpha(\chi) + \alpha(\eta) - \alpha(\chi + \eta)$ is in $\mathfrak{p}_A^+$.

We now show that

$$c\alpha(\chi) - \alpha(c\chi)$$

is in $\mathfrak{p}_A^+$.

Suppose, on the contrary, that for some X in $\mathbf{F}$,

$$c\alpha(\chi)(X) < \alpha(c\chi)(X) .$$

Obviously $c > 0$. There is a γ in $M \cap \mathfrak{p}_A^+$ such that $c\chi - \gamma$ is in $\mathfrak{p}_A^+$ and

$$c\boldsymbol{\alpha}(\chi)(X) < \gamma(X) .$$

Now $\chi - (1/c)\gamma$ is in $\mathfrak{p}_A$ and $(1/c)\gamma$ is in $M \cap \mathfrak{p}_A^+$, which implies that $\boldsymbol{\alpha}(\chi) - (1/c)\gamma$ is in $\mathfrak{p}_A$, so that

$$(1/c)\gamma(X) \leqslant \boldsymbol{\alpha}(\chi)(X) ,$$

so that

$$\gamma(X) \leqslant c\boldsymbol{\alpha}(\chi)(X) ,$$

a contradiction. Therefore $c\boldsymbol{\alpha}(\chi) - \boldsymbol{\alpha}(c\chi)$ is in $\mathfrak{p}_A^+$.

Before proving Theorem 6.1, we gain degress to show, as asserted in the introduction, that Theorem 6.1 is indeed a generalization of the second inequality of Theorem A.1.

We first observe (see [2], p. 142) that if each of ι and ψ is in $\mathfrak{p}_A^+$ and

$$0 = \int_U \min\{\iota(I), \psi(I)\} ,$$

then

$$0 = \int_U \min\{\iota(I), K\psi(I)\} , \ 0 \leqslant K .$$

This implies, by the second inequality of Theorem A.1, that if χ is in $\mathfrak{p}_A$, η is in A_μ, and $1 \leqslant K$, then

$$0 = \int_U \min\{|\chi(I) - \boldsymbol{\delta}(\chi)(I)|, K\mu(I)\},$$

and

$$0 < \int_U \min\{|\chi(I) - \eta(I)|, \mu(I)\} \leqslant \int_U \min\{|\chi(I) - \eta(I)|, K\mu(I)\},$$

so that

$$\int_U |\boldsymbol{\delta}(\chi - \boldsymbol{\delta}(\chi))(I)| = \boldsymbol{\lambda}^*(\chi - \boldsymbol{\delta}(\chi))(U) = 0 < \boldsymbol{\lambda}^*(\chi - \eta)(U) =$$
$$\int_U |\boldsymbol{\delta}(\chi - \eta)(I)| ,$$

which is the conclusion of Theorem 6.1 with $\boldsymbol{\alpha} = \boldsymbol{\delta}$.

We now prove Theorem 6.1.

Proof of Theorem 6.1. First, suppose η is in $M \cap \mathfrak{p}_A^+$ and χ is in $\mathfrak{p}_A^+$ and

$$\eta \neq \alpha(\chi) .$$

We show that

$$0 < \int |\alpha(\chi - \eta)(I)| .$$

Let β denote $\int \max\{\eta, \alpha(\chi)\}$. Each of β and $\beta - \alpha(\chi)$ is in $M \cap \mathfrak{p}_A^+$. Now suppose

$$\int_U |\alpha(\chi - \eta)(I)| = 0 .$$

Then

$$0 = \alpha\left(\int |\chi - \eta|\right)(U) = \alpha\left(\int \max\{\chi, \eta\} - \int \min\{\chi, \eta\}\right)(U) \geqslant$$

$$\geqslant \alpha\left(\left[\int \max\{\chi, \eta\}\right] - \alpha(\chi)\right)(U) \geqslant \alpha\left(\left[\int \max\{\alpha(\chi), \eta\}\right] - \alpha(\chi)\right)(U) ,$$

so that

$$\int \max\{\alpha(\chi), \eta\} = \alpha(\chi) ,$$

so that $\alpha(\chi) - \eta$ is in $M \cap \mathfrak{p}_A^+$, which implies that $\chi - \eta$ is in $\mathfrak{p}_A^+$, so that

$$0 = \alpha(\chi - \eta)(U) \geqslant \alpha(\alpha(\chi) - \eta)(U) ,$$

which implies that

$$\alpha(\chi) = \eta ,$$

a contradiction. Therefore

$$0 < \int_U |\alpha[(\chi - \eta)](I)| .$$

Now suppose χ is in $\mathfrak{p}_A$.
We show that

$$0 = \int_U |\alpha[\chi - \alpha(\chi)](I)| .$$

Let χ^* denote $\int|\chi|$. Now

$$\int_U |\alpha[\chi - \alpha(\chi)](I)| = \alpha\left(\int |\chi - \alpha(\chi)|\right)(U) =$$

$$= \alpha\left(\int |\tau(\chi)|\,|\chi - \tau(\chi)\alpha(\chi^*)|\right)(U) = \alpha\left(\int |\tau(\chi)\chi - \tau(\chi)^2\alpha(\chi^*)|\right)(U) =$$

$$= \alpha\left(\int \big||\chi| - \alpha(\chi^*)\big|\right)(U) = \alpha\left(\int |\chi^* - \alpha(\chi^*)|\right)(U) =$$

$$= \alpha(\chi^* - \alpha(\chi^*))(U) = 0\,,$$

by Theorem 5.1.

Now suppose η is in M and

$$\eta \neq \alpha(\chi)\,.$$

We see that $\int|\eta|$ is in $M \cap \mathfrak{p}_A^+$. Also, either

$$\int|\eta| = \int|\alpha(\chi)|$$

or not.

Suppose

$$\int|\eta| \neq \int|\alpha(\chi)|\,.$$

Then

$$\int_U |\alpha(\chi - \eta)(I)| = \alpha\left(\int |\chi - \eta|\right)(U) \geqslant \alpha\left(\int \big||\chi| - |\eta|\big|\right)(U) =$$

$$= \alpha\left(\int \left|\left[\int|\chi|\right] - \int|\eta|\right|\right)(U) > 0\,,$$

by the first portion of this proof.

Now suppose

$$\int|\eta| = \int|\alpha(\chi)|\,.$$

We see that

$$\int_U \tau(\chi)(I)\,\eta(I)$$

exists and $\alpha(\chi^*) - \int \tau(\chi)\eta$ is in $\mathfrak{p}_A^+$. Therefore

$$\alpha(\int |\chi - \eta|)(U) = \alpha(\int |\tau(\chi)||\chi - \eta|)(U) =$$

$$= \alpha(\int |\tau(\chi)\chi - \tau(\chi)\eta|)(U) = \alpha(\int |\chi^* - \alpha(\chi^*) + \alpha(\chi^*) - \int \tau(\chi)\eta|)(U) \geqslant$$

$$\geqslant \alpha(\int |\alpha(\chi^*) - \tau(\chi)\eta|)(U) = \alpha(\int |\tau(\chi)\alpha(\chi^*) - \tau(\chi)^2\eta|)(U) =$$

$$= \alpha(\int |\tau(\chi)\alpha(\chi^*) - \eta|)(U) = \alpha(\int |\alpha(\chi) - \eta|)(U) > 0\,,$$

since $\alpha(\chi) - \eta$ is in M and

$$\int_U |\alpha(\chi)(I) - \eta(I)| > 0\,.$$

Therefore

$$\int_U |\alpha(\chi - \eta)(I)| > 0\,.$$

We now prove Corollary 6.1.

Corollary 6.1. α is linear.

Proof. Suppose each of χ and η is in $\mathfrak{p}_A$ and c is a number. Suppose

$$\alpha(\chi + \eta) \neq \alpha(\chi) + \alpha(\eta)\,.$$

Then, since $\alpha(\chi) + \alpha(\eta)$ is in M, it follows from Lemma 6.1 and Theorem 6.1 that

$$0 < \alpha(\int |\chi + \eta - [\alpha(\chi) + \alpha(\eta)]|)(U) \leqslant$$

$$\leqslant \alpha(\int |\chi - \alpha(\chi)| + \int |\eta - \alpha(\eta)|)(U) \leqslant \alpha(\int |\chi - \alpha(\chi)|)(U) +$$

$$+ \alpha(\int |\eta - \alpha(\eta)|)(U) = 0 + 0 = 0\,,$$

a contradiction. Therefore

$$\alpha(\chi + \eta) = \alpha(\chi) + \alpha(\eta)\,.$$

Now suppose

$$c\alpha(\chi) \neq \alpha(c\chi)\,.$$

Then, since $c\alpha(\chi)$ is in M, it again follows from Lemma 6.1 and Theorem 6.1 that

$$0 < \alpha\left(\int |c\chi - c\alpha(\chi)|\right)(U) = \alpha\left(|c|\int|\chi - \alpha(\chi)|\right)(U) \leqslant$$

$$\leqslant |c|\alpha\left(\int |\chi - \alpha(\chi)|\right)(U) = |c|0 = 0\,,$$

a contradiction. Therefore

$$c\alpha(\chi) = \alpha(c\chi)\,.$$

Therefore α is linear.

7. - Examples of C-sets.

In this section we discuss some examples of C-sets other than A_μ, where μ is in $\mathfrak{p}_A^+$. We leave the proof that they are C-sets to the reader. We will state, in each case, whether or not it is a linear space, but again leave the proof of the assertion to the reader.

Example 7.1. The set of all « continuous » elements of $\mathfrak{p}_A$, *i.e.*, the set to which χ belongs iff χ is in $\mathfrak{p}_A$ and for each $c > 0$ there is a subdivision $\mathfrak{D}$ of U such that if I is in a refinement of $\mathfrak{D}$, then

$$\int_I |\chi(J)| < c\,.$$

This is a C-set and a linear space.

Example 7.2. For μ in $\mathfrak{p}_A^+$ and $0 \leqslant K$, the set of all χ in $\mathfrak{p}_A$ such that for all I in $\mathbf{F}$,

$$\int_I |\chi(J)| \leqslant K\mu(I)\,.$$

This is a C-set, but not necessarily a linear space.

Example 7.3. For β in $\mathfrak{p}_B$, the set of all χ in $\mathfrak{p}_A$ such that

$$\int_U \beta(I)\,\chi(I)$$

exists. This is a C-set and a linear space.

Example 7.4. For β in $\mathfrak{p}_B$, the set of all χ in $\mathfrak{p}_A$ such that

$$\int_V \beta(I)\,\chi(I)$$

exists and is 0 for all V in **F**. This is a C-set and a linear space.

Example 7.5. If M is a C-set, then the set of all χ in $\mathfrak{p}_A$ such that

$$\alpha(\chi) = 0$$

(where α is, of course, the « nearest point transformation » associated with M) is a C-set and a linear space.

Example 7.6. If G is a collection of C-sets, then $\cap\, G$ is a C-set, although, of course, not necessarily a linear space.

References.

[1] W. D. L. Appling, *Set functions and extremal problem in absolute continuity*, Rend. Circ. Mat. Palermo (2) **14** (1965), 256-266.

[2] W. D. L. Appling, *Refinement-Unbounded set functions and Absolute continuity*, Math. Zeitschr. **92** (1966), 140-145.

[3] W. D. L. Appling, *Concerning a class of linear transformations*, J. London Math. Soc. **44** (1969), 385-396.

[4] W. D. L. Appling, *Concerning set functions with finite ranges*, Matematica (33) **10** (1968), 211-222.

[5] A. Kolmogoroff, *Untersuchungen uber den Integralbegriff*, Math. Ann. **103** (1930), 654-696.

* * *