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MOHD. NASIM KHAN (*)

A Continuous Ring in which Every Large Right Ideal is two Sided. (**)

The concept of q-ring was introduct by JAIN, MOHAMED and SINGH in [1]. They defined that a ring R is to be right (left) q-ring if every right (left) ideal of R is quasi-injective and proved the following theorem:

THEOREM 1. *The following are equivalent:*

(i) *R is a right q-ring.*

(ii) *R is right self-injective and every right ideal of R is of the form eI, e is an idempotent and I is a two-sided ideal in R.*

(iii) *R is right self-injective and every large right ideal of R is two-sided.*

UTUMI defined in [5], the concept of a continuous ring as a generalization of the self-injective ring as follows; A ring R is said to be right continuous if it satisfies the following conditions:

(i) For any right ideal A there is an idempotent e such that eR is an essential extension of A.

(ii) If fR, $f = f^2$ is isomorphic to a right ideal B, then B is also generated by an idempotent.

In this Note, a ring R is said to be right (left) (cq)-ring if it is right (left) continuous and each of its large right (left) ideals is two-sided. In the last,

(*) Indirizzo: Department of Mathematics and Statistics, Aligarh Muslim University, Aligarh (U.P.), India.

(**) Ricevuto: 11-IV-1972.

we have shown by an example that a right (cq)-ring need not be a right q-ring. However it is clear from Theorem 1 that every right q-ring is a right (cq)-ring. Here we attempt to generalize some results for q-rings to (cq)-rings.

For any ring R, R^{Δ}, $J(R)$ and $B(R)$ will denote the right singular ideal, the JACOBSON radical and prime radical respectively and essentially the terminologies are same as in [**1**].

Throughout this paper, unless otherwise stated, we assume that every ring has unity $1 \neq 0$.

It can be ssen easily that the following results have the same proofs as for q-rings in [**1**].

Theorem 2. *Let $n > 1$ be an integer. Then R_n is a right (cq)-ring if and only if R is semi-simple artinian.*

Theorem 3. *A simple ring is a right (cq)-ring if and only if R is artinian.*

Since in a right continuous ring R, $R^{\Delta} = J(R)$ ([**5**] Lemma 4.1) so it can be easily seen on the same lines as ([**1**] Lemma 2.8) that following holds.

Lemma 1. *Let R be a right (cq)-ring, then $B(R)$ is essential in $J(R)$, as right ideals.*

Theorem 4. *A right (cq)-ring is regular if and only if it is semi-prime.*

Proof. Let R be a right (cq)-ring. Then by UTUMI ([**5**] Lemma 4.1) $R/J(R)$ is regular. Since $B(R) \subset' J(R)$ by Lemma 1. We see that $J(R) = 0$ if $B(R) = 0$. Hence R is regular if and only if R is semi-prime.

Theorem 5. *Let V be a vector space our some division ring D and $R = Hom_D(V, V)$. Then R is a right (cq)-ring if and only if V is finite dimensional.*

Proof. If V is finite dimensional, then R is simple artinian so trivially R is a (cq)-ring.

Let V be not of finite dimensional, then $V \cong V \oplus V$. This yields $R \cong R_2$ so if R were a (cq)-ring then R_2 is also a (cq)-ring. Hence by Theorem 2 R is simple artinian so V must be of finite dimension.

The proof of the following Lemmas are the same as of those for q-rings.

Lemma 2. *A semi-prime right (cq)-ring with zero socle is strongly regular.*

Lemma 3. *A prime right (cq)-ring has non-zero socle.*

Now we prove the following:

Theorem 6. *A prime right (cq)-ring is simple artinian.*

Proof. Let R be a prime right (cq)-ring, then by Lemma 2, R has a non-zero socle so R is primitive. Hence by UTUMI ([5], Theorem 7.9) R is right self-injective ring. Since by definition of (cq)-ring, every large right ideal is two-sided, by Theorem 1, R is right q-ring.

Thus by ([1], Theorem 2.13) R is simple artinian.

Finally we have the following structure theorem:

Theorem 7. *If R is a semi-prime right (cq)-ring, then $R = A \oplus B$ where A is a d.u.o. ring and B is semi-simple artinian.*

Proof. Let R be a semi-prime right (cq)-ring then R is regular by Theorem 4 and $L(R)$ the lattice of all right ideals of R is complete. So by UTUMI ([3], Corollary of Theorem 4) for every positive integer n there is the decomposition $R = R_n \oplus R_n'$ such that R_n is an ideal of index $\leqslant n$ and R_n' is an ideal not containing any ideal of index $\leqslant n$. In particular if we put $n = 1$ then $R = R_1 \oplus R_1'$ where R_1 is, if non-zero, an ideal of index 1 and R_1' is an idean not containing any ideal of index 1. So by UTUMI ([3], Theorem 3) R_1' is right self-injective and hence R_1' is a right q-ring. Thus by ([1], Theorem 2.19). $R_1' = A' \oplus B$ where A' is a d.u.o. ring and B is semi-simple artinian. Again since R_1 is an ideal of index 1 so every idempotent of R_1 is central and hence R_1 is a d.u.o. ring. Thus we have $R = A \oplus B$ such that $A = R_1 \oplus A'$ is a d.u.o. ring and B is a semi-simple artinian ring.

In the above structure theorem we have $R = A \oplus B$ such that A is a d.u.o. right (cq)-ring and B is semi-simple artinian. Since every left ideal of B is a direct summand and every one sided ideal of A is two-sided so the following can be easily proved:

Lemma 4. *If a semi-prime ring R is right (cq)-ring then it is also left (cq)-ring.*

Example. Let $\{K_\alpha\}_{\alpha \in N}$ be any infinite family of fields such that each K_α has a proper subfield say F_α. Let $S = \Pi K_\alpha$ and T be the ring of all those elements in S which have all except a finite number of components in F_α. Then T is continuous but not self-injective by UTUMI ([3], Example 3). Since T is commutative so each of its one sided ideal is two-sided. Hence Π is a (cq)-ring but not a q-ring.

Remark. One can study those rings in which every large right ideal is two-sided. One can immediately see that Theorem 2, 3 and 4 all hold for such type of rings. Further notice that Lemma 1 does not hold for such rings. For example consider any domain D having a unique maximal ideal M. Then $B(D) = 0$ and $J(D) = M$ so $B(D)$ is not large in $J(D)$.

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