Riv. Mat. Univ. Parma (3) 3 (1974), 107-131

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A Boundary Value Problem for Quasi Linear Hyperbolic Systems. (**)

Ad ANTONIO MAMBRIANI per il suo 75° compleanno

1. - Introduction.

In the present paper we take into consideration the following canonic form of quasilinear hyperbolic systems

$$(1.1)\qquad \partial z_i/\partial x + \sum_{k=1}^{r} \varrho_{ik}(x, y, z)\, \partial z_i/\partial y_k = f_i(x, y, z)\,, \qquad i = 1, \dots, m\,,$$

$$z(x, y) = (z_1, \dots, z_m)\,,\; y = (y_1, \dots, y_r)\,,$$

in a slab $D_a = I_a \times E^r$, $I_a = [x \,|\, 0 \leqslant x \leqslant a]$. Instead of usual CAUCHY data at $x = 0$, we shall take into consideration here more general types of boundary data (I, II, III below).

I. For instance, we may assume that certain functions $\psi_i(y)$, $y \in E^r$, $i = 1, \dots, m$, and an integer m', $0 \leqslant m' \leqslant m$, are assigned, and we may request that

$$z_i(0, y) = \psi_i(y)\,, \qquad i = 1, \dots, m'\,,\; y \in E^r\,,$$

$$z_i(a, y) = \psi_i(y)\,, \qquad i = m' + 1, \dots, m\,,\; y \in E^r\,.$$

For $m' = m$ (as well as for $m' = 0$) we have the usual CAUCHY problem.

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(**) This research was partially supported by AFOSR Research Project 71-2122 at the University of Michigan. Ricevuto: 9-IX-1974.

II. More generally, we may assume that certain numbers a_i, $0 \leqslant a_i \leqslant a$, and functions $\psi_i(y)$, $y \in E^r$, $i = 1, \ldots, m$, are assigned, and we may request that

$$z_i(a_i, y) = \psi_i(y)\,, \qquad y \in E^r\,,\ i = 1, \ldots, m\,.$$

III. In a more general setting, we may assume that certain numbers a_i, $0 \leqslant a_i \leqslant a$, functions $\psi_i(y)$, $y \in E^r$, $i = 1, \ldots, m$, and an $m \times m$ matrix $[b_{ij}(y), i, j = 1, \ldots, m]$, $y \in E^r$, are assigned, and we may request that

$$\sum_{j=1}^{m} b_{ij}(y)\, z_j(a_i, y) = \psi_i(y)\,, \qquad y \in E^r\,,\ i = 1, \ldots, m\,.$$

If $[b_{ij}]$ is the identity matrix, then this boundary condition III reduces to II. If furthermore, $a_i = 0$ for $i = 1, \ldots, m'$, $a_i = a$ for $i = m' + 1, \ldots, m$, $0 \leqslant m' \leqslant m$, then we have problem I.

In the present paper we prove a theorem of existence, uniqueness and continuous dependence on the data for the hyperbolic system (1.1) with general boundary conditions III, for matrices $[b_{ij}]$ whose main diagonal $[b_{ii}, i = 1, \ldots, m]$ is dominant. This certainly includes both problems I and II as well as the CAUCHY problem, for all of which the matrix $[b_{ij}]$ is the identity matrix.

An $m \times m$ matrix $[b_{ij}]$ is usually said to have dominant main diagonal if $\sum_{j \neq i} |b_{ij}| < |b_{ii}|$, $i = 1, \ldots, m$. It is easier for us to assume simply that

$$b_{ij}(y) = \delta_{ij} + \tilde{b}_{ij}(y)\,, \qquad i, j = 1, \ldots, m\,,\ y \in E^r\,,$$

with $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$, and that

$$\sum_{j=i}^{m} |\tilde{b}_{ij}(y)| \leqslant \sigma < 1\,, \qquad i = 1, \ldots, m\,,\ y \in E^r\,,$$

where σ is a fixed number, $0 \leqslant \sigma < 1$. Thus, when $[b_{ij}]$ is the identity matrix as in problems I and II and the CAUCHY problem, all $\tilde{b}_{ij}(y) = 0$, $i, j = 1, \ldots, m$, and $\sigma = 0$.

Our existence theorem reduces to a well known existence theorem [2] for the CAUCHY problem.

Since we obtain the solution as the fixed point of transformations which are contractions in the uniform topology, the usual iterative scheme is uniformly convergent to the unique solution.

In 2 we give a new proof of the existence theorem for the CAUCHY problem, proof based on BANACH's fixed point theorem. In 3 we prove the existence theorem for boundary conditions III (thus, including boundary conditions I and II). The proof is also based on BANACH's fixed point theorem, and the precise estimates obtained in **2**. In 4 we give a new proof of a chain rule differentiation statement used in **2**.

Added in proof. A bibliography on the subject and further work are presented in the paper: L. CESARI, *A boundary value problem for quasilinear hyperbolic systems in the Schauder canonic form*, Annali Scuola Normale Sup. Pisa, Ser. 4, Vol. 1, pp. 311-358, (1974).

2. - The existence theorem for the Cauchy problem.

We denote by x a scalar, by y an r-vector $y=(y_1, \dots, y_r)$, and by z an m-vector $z=(z_1, \dots, z_m)$. We take in E^r and E^m the norms $|y|=\max_k |y_k|$, and $z=\max_i |z_i|$.

We consider hyperbolic systems of the canonic form

$$\partial z_i/\partial x + \sum_{k=1}^{r} \varrho_{ik}(x, y, z)\, \partial z_i/\partial y_k = f_i(x, y, z)\,, \qquad i=1, \dots, m\,,\ (x, y)\in D_a\,, \tag{2.1}$$

with CAUCHY data

$$z_i(0, y)=\varphi_i(y)\,, \qquad i=1, \dots, m\,,\ y\in E^r\,, \tag{2.2}$$

where $z(x, y)=(z_1, \dots, z_m)$, $y=(y_1, \dots, y_r)$. We seek existence and uniqueness of the solution $z(x, y)$ in an infinite slab

$$D_a = I_a \times E^r = [(x, y)\,|\,0\leqslant x\leqslant a,\ y\in E^r]\subset E^{r+1}\,, \qquad I_a=[x\,|\,0\leqslant x\leqslant a]\,.$$

Theorem I. (*An existence theorem for the Cauchy problem*). *Let $m(x)$, $h(x)$, $l(x)$, $0\leqslant x\leqslant a_0$, be nonnegative functions, $m, h, l\in L_1[0, a_0]$. Let Λ, ω, Ω be given numbers, $0<\omega<\Omega$, and let Ω also denote the interval $[-\Omega, \Omega]^m$ in E^m.*

Let $\varrho_{ik}(x, y, z)$, $f_i(x, y, z)$, $i=i, \dots, m$, $k=1, \dots, r$, be given functions defined on $D_{a_0}\times\Omega$, all measurable in x for every (y, z), and continuous in (y, z) for every x, such that, for all (x, y, z), $(x, \bar y, \bar z)\in D_{a_0}\times\Omega$ and $i=1, \dots, m$, $k=1, \dots, r$, we have

$$|\varrho_{ik}(x, y, z)|\leqslant m(x)\,, \qquad |f_i(x, y, z)|\leqslant h(x)\,, \tag{2.3}$$

$$|\varrho_{ik}(x, y, z)-\varrho_{ik}(x, \bar y, \bar z)|\leqslant l(x)[|y-\bar y|+|z-\bar z|]\,, \tag{2.4}$$

$$|f_i(x, y, z)-f_i(x, \bar y, \bar z)|\leqslant l(x)[|y-\bar y|+|z-\bar z|]\,. \tag{2.5}$$

Let $\varphi(y) = (\varphi_1, \ldots, \varphi_m)$, $y \in E^r$, *be given functions defined in* E^r *such that, for all* y, $\bar{y} \in E^r$, *and* $i = 1, \ldots, m$, *we have*

$$|\varphi_i(y)| \leqslant \omega\,, \qquad |\varphi_i(y) - \varphi_i(\bar{y})| \leqslant \Lambda |y - \bar{y}|\,. \tag{2.6}$$

Then, there is a number a, $0 < a \leqslant a_0$, *and continuous functions* $z(x, y) = (z_1, \ldots, z_m)$, $(x, y) \in D_a$, *absolutely continuous in* x *for every* y, *uniformly Lipschitzian in* y *for every* x, *satisfying*

$$-\Omega \leqslant z_i(x, y) \leqslant \Omega\,, \qquad (x, y) \in D_a\,,\ i = 1, \ldots, m\,, \tag{2.7}$$

satisfying (2.1) *a.e. in* D_a, *and* (2.2) *everywhere in* E^r. *This solution* z *is unique and depends continuously on* φ *(in the classes* $\mathscr{K}_1$, $\mathscr{K}_0$ *to be stated in the proof).*

Proof. The proof is divided into parts (*a*), ..., (*e*).

(a) *Choice of constants.* For every a, $0 \leqslant a \leqslant a_0$, let $M_a = \int_0^a m(\alpha)\,d\alpha$, $H_a = \int_0^a h(\alpha)\,d\alpha$, $L_a = \int_0^a l(\alpha)\,d\alpha$. Let us choose constants p, Q, k with $0 < p < 1$, $Q > (1 + p)\Lambda$, $0 < k < 1$, and let us take a, $0 < a \leqslant a_0$, sufficiently small so that

$$\omega + H_a \leqslant \Omega\,, \qquad L_a(1 + Q) \leqslant k\,, \tag{2.8}$$

$$L_a(1 + p)(1 + Q) < p\,, \qquad (1 + p)(\Lambda + L_a(1 + Q)) \leqslant Q\,, \tag{2.9}$$

$$L_a(1 + \lambda(\Lambda + L_a(1 + Q))) < 1\,, \tag{2.10}$$

with λ, ν denoting the constants, $\lambda > 1$, $0 < \nu < 1$,

$$\lambda = (1 - L_a(1 + Q))^{-1}\,, \qquad \nu = L_a(1 + \lambda(\Lambda + L_a(1 + Q)))\,.$$

(b) *The classes* $\mathscr{K}_0$ *and* $\mathscr{K}_1$. We denote by D_a and Δ_a the regions

$$D_a = I_a \times E^r = [(x, y)\,|\,0 \leqslant x \leqslant a,\ y \in E^r] \subset E^{r+1}\,,$$

$$\Delta_a = I_a \times I_a \times E^r = [(\xi, x, y)\,|\,0 \leqslant \xi \leqslant a,\ 0 \leqslant x \leqslant a,\ y \in E^r] \subset E^{r+2}\,.$$

Let $\mathscr{K}_0$ be the set of all systems

$$g = [g_{ik}(\xi; x, y),\ i = 1, \ldots, m,\ k = 1, \ldots, r]\,, \tag{2.11}$$

of continuous functions g_{ik} in Δ_a satisfying the following conditions

$$g_{ik}(x; x, y) = y_k \qquad \text{for all } (x, y) \in D_a, \tag{2.12}$$

$$|g_{ik}(\xi; x, y) - g_{ik}(\bar{\xi}; x, y)| \leq \left|\int_{\xi}^{\bar{\xi}} m(\alpha)\,\mathrm{d}\alpha\right|, \tag{2.13}$$

$$|g_{ik}(\xi; x, y) - g_{ik}(\xi; x, \bar{y}) - y_k + \bar{y}_k| \leq p\,|y - \bar{y}| \tag{2.14}$$

$$\text{for all } (\xi; x, y),\ (\bar{\xi}; x, y),\ (\xi; x, \bar{y}) \in \Delta_a.$$

Thus, each function g_{ik} is absolutely continuous in ξ for every (x, y), and we have

$$|\partial g_{ik}(\xi; x, y)/\partial\xi| \leq m(\xi),$$

a.e. in Δ_a, $i = 1, \ldots, m$, $k = 1, \ldots, r$. For every $i = 1, \ldots, m$, we denote by $\check{g}_i(\xi; x, y)$ the r-vector $\check{g}_i(\xi; x, y) = (g_{ik}, k = 1, \ldots, r)$. We shall denote by $\tilde{\mathscr{K}}_0$ the set of all systems

$$h = [h_{ik}(\xi; x, y),\ i = 1, \ldots, m,\ k = 1, \ldots, r],$$

with $h_{ik} = g_{ik}(\xi; x, y) - y_k$, $(\xi; x, y) \in \Delta_a$, $g = [g_{ik}] \in \mathscr{K}_0$. Thus, if $\check{h}_i = [h_{ik}, k = 1, \ldots, r]$, we have $\check{h}_i = \check{g}_i(\xi; x, y) - y$, $(\xi, x, y) \in \Delta_a$, $g = [g_{ik}] \in \mathscr{K}_0$. Then relations (2.12), (2.14) become

$$h_{ik}(x; x, y) = 0 \qquad \text{for all } (x, y) \in \Delta_a,$$

$$|h_{ik}(\xi; x, y) - h_{ik}(\bar{\xi}; x, y)| \leq \left|\int_{\xi}^{\bar{\xi}} m(\alpha)\,d\alpha\right|,$$

$$|h_{ik}(\xi; x, y) - h_{ik}(\xi; x, \bar{y})| \leq p\,|y - \bar{y}|$$

for all $(\xi; x, y)$, $(\bar{\xi}; x, y)$, $(\xi; x, \bar{y}) \in \Delta_a$.

Thus, for $(\xi; x, y) \in \Delta_a$ we have

$$|h_{ik}(\xi; x, y)| = |h_{ik}(x; x, y) + [h_{ik}(\xi; x, y) - h_{ik}(x; x, y)]| \leq M_a,$$

that is, the functions h_{ik} are uniformly bounded in Δ_a. Also

$$|\check{h}_i(\xi; x, y) - \check{h}_i(\xi; x, \bar{y})| = \mathrm{Max}_k |h_{ik}(\xi; x, y) - h_{ik}(\xi; x, \bar{y})| \leq p\,|y - \bar{y}|. \tag{2.15}$$

Finally, for the r-vector functions $\breve{g}_i(\xi; x, y)$ we also have

$$(2.16)\qquad \begin{cases} \breve{g}_i(x; x, y) = y \\ |g_{ik}(\xi; x, y) - g_{ik}(\xi; x, \bar{y})| \leqslant (1+p)\,|y - \bar{y}|\,, \\ |g_i(\xi; x, y) - \breve{g}_i(\xi; x, \bar{y})| \leqslant (1+p)\,|y - \bar{y}|\,. \end{cases}$$

Note that $\mathscr{K}_0$ is a subset of the BANACH space $\big(C(\Delta_a) \cap L_\infty(\Delta_a)\big)^{mr}$ with norm

$$\|h\| = \max_i \|\breve{h}\|\,, \qquad \breve{h}_i = [h_{ik},\ k = 1, \dots, r]\,,$$

$$\|\breve{h}_i\| = \max_k \|h_{ik}\|\,, \qquad \|h_{ik}\| = \operatorname{Sup}_{\Delta_a} |h_{ik}(\xi; x, y)|\,.$$

We also consider the set $\mathscr{K}_1$ of all systems

$$(2.17)\qquad z = [z_i(x, y),\ i = 1, \dots, m]$$

of continuous bounded functions z_i in D_a satisfying the following conditions

$$(2.18)\qquad -\Omega \leqslant z_i(x, y) \leqslant \Omega\,,$$

$$(2.19)\qquad |z_i(x, y) - z_i(x, \bar{y})| \leqslant Q\,|y - \bar{y}|\,,$$

for all (x, y), $(x, \bar{y}) \in D_a$, $i = 1, \dots, m$. Thus, we have also

$$|z(x, y)| \leqslant \Omega\,, \qquad |z(x, y) - z(x, \bar{y})| \leqslant Q\,|y - \bar{y}|$$

for all (x, y), $(x, \bar{y}) \in D_a$. Here $\mathscr{K}_1$ is a subset of the Banach space $\big(C(D_a) \cap \cap L_\infty(D_a)\big)^m$ with norm

$$\|z\| = \max_i \|z_i\|\,, \qquad \|z_i\| = \operatorname{Sup}_{D_a} |z_i(x, y)|\,.$$

(c) *The transformation* T_z. For every fixed $z \in \mathscr{K}_1$, let us consider the transformation T_z defined on $\mathscr{K}_0$, say $G = T_z g$, $g \in \mathscr{K}_0$, or $[g_{ik}] \to [G_{ik}]$, by taking

$$(2.20)\qquad \begin{aligned} &G_{ik}(\xi; x, y) = y_k - \int_\xi^x \varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y)))\, d\alpha\,, \\ &(\xi; x, y) \in \Delta_a = I_a \times I_a \times E^r\,,\ i = 1, \dots, m\,,\ k = 1, \dots, r\,. \end{aligned}$$

Note that the functions G_{ik} are obviously continuous, and that

$$G_{ik}(x;\, x,\, y) = y_k \qquad \text{for all } (x, y) \in I_a \times E^r\,; \tag{2.21}$$

$$|G_{ik}(\xi;\, x,\, y) - G_{ik}(\bar{\xi};\, x,\, y)| \leq \Big|\int_{\xi}^{\bar{\xi}} m(\alpha)\, d\alpha\Big|\,; \tag{2.22}$$

$$\left\{\begin{aligned} & |\,G_{ik}(\xi;\, x,\, y) - G_{ik}(\xi;\, x,\, \bar{y}) - y_k + \bar{y}_k| \leq \\ & \leq \Big|\int_{\xi}^{x} |\,\varrho_{ik}(\alpha, \breve{g}_k(\alpha;\, x,\, y), z(\alpha, \breve{g}_i(\alpha;\, x,\, y))) - \\ & - \varrho_{ik}(\alpha, \breve{g}_i(\alpha;\, x,\, \bar{y}), z(\alpha, \breve{g}_i(\alpha;\, x,\, \bar{y})))\,|\,\mathrm{d}\alpha\,| \leq \\ & \leq \Big|\int_{\xi}^{x} l(\alpha)(1 + Q)\,|\breve{g}_i(\alpha;\, x,\, y) - \breve{g}_i(\alpha;\, x,\, \bar{y})\,|\,\mathrm{d}\alpha \leq \\ & \leq L_a(1 + p)(1 + Q)\,|y - \bar{y}\,| \leq p\,|y - \bar{y}\,| \end{aligned}\right. \tag{2.23}$$

for all $(\xi;\, x,\, y)$, $(\bar{\xi};\, x,\, y)$, $(\xi;\, x,\, \bar{y}) \in \Delta_a$, $i = 1, \ldots, m$, $k = 1, \ldots, r$. We have used here inequalities (2.3), (2.4), (2.16), (2.19) and (2.9).

By comparison of (2.21), (2.22), (2.23) with (2.12), (2.13), (2.14) we conclude that $G = T_z g$ belongs to $\mathscr{K}_0$. In other words, for every $z \in \mathscr{K}_1$, the transformation T_z defined above is a map $T_z\colon \mathscr{K}_0 \to \mathscr{K}_0$. Considering the differences $h_{ik} = g_{ik} - y_k$, we may well think of T_z as a map $T_z\colon \tilde{\mathscr{K}}_0 \to \tilde{\mathscr{K}}_0$ with $\tilde{\mathscr{K}}_0$ a subset of a BANACH space. Let us prove that $\tilde{T}_z\colon \tilde{\mathscr{K}}_0 \to \tilde{\mathscr{K}}_0$ is a contraction. Indeed, if $g, g' \in \mathscr{K}_0$, $G = T_z g$, $G' = T_z g'$, and h, h', H, H' are the corresponding elements in $\tilde{\mathscr{K}}_0$, then

$$\begin{aligned} |H_{ik} - H'_{ik}| & \leq \Big|\int_{\xi}^{x} |\,\varrho_{ik}(\alpha, \breve{g}_i(\alpha;\, x,\, y), z(\alpha, \breve{g}_i(\alpha;\, x,\, y))) - \\ & - \varrho_{ik}\,(\alpha, \breve{g}'_i(\alpha;\, x,\, y), z(\alpha, \breve{g}_i(\alpha;\, x,\, y)))\,|\,\mathrm{d}\alpha\,| \leq \\ & \leq \int_{0}^{a} l(\alpha)[\,|\,\breve{g}_i(\alpha;\, x,\, y) - \breve{g}'_i(\alpha;\, x,\, y)\,| + \\ & + |z(\alpha, \breve{g}_i(\alpha;\, x,\, y)) - z(\alpha, \breve{g}_i(\alpha;\, x,\, y))\,|\,]\,\mathrm{d}\alpha \leq \\ & \leq L_a(1 + Q)\,\sup_{\Delta_a} |\,\breve{g}_i(\alpha;\, x,\, y) - \breve{g}'_i(\alpha;\, x,\, y)\,| = \\ & = L_a(1 + Q)\,\sup_{\Delta_a} |\,\breve{h}_i(\alpha;\, x,\, y) - \breve{h}'_i(\alpha;\, x,\, y)\,| \leq \\ & \leq L_a(1 + Q)\,\|h_i - h'_i\|\,. \end{aligned}$$

By the definition of norm $\|h\|$ we obtain, by force of (2.8),

$$\|H - H'\| \leqslant L_a(1+Q)\|h - h'\| \leqslant k\|h - h'\| ,$$

where $k < 1$. Thus, for every $z \in \mathscr{K}_1$, the map $\tilde{T}_z\colon \tilde{\mathscr{K}}_0 \to \tilde{\mathscr{K}}_0$ is a contraction of constant $k < 1$.

We conclude that $\tilde{T}_z\colon \tilde{\mathscr{K}}_0 \to \tilde{\mathscr{K}}_0$ has a fixed point $h \in \tilde{\mathscr{K}}_0$, and the corresponding element $g \in \mathscr{K}_0$ is a fixed point of the transformation $T_z\colon \mathscr{K}_0 \to \mathscr{K}_0$. We shall denote this fixed element by $g = g[z] \in \mathscr{K}_0$, or $g(\xi; x, y) = [g_{ik},\ i = 1, \dots, m,\ k = 1, \dots, r]$, and $g[z]$ satisfies the integral equations

$$(2.24) \qquad g_{ik}(\xi; x, y) = y_k - \int_\xi^x \varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y)))\, \mathrm{d}\alpha ,$$
$$k = 1, \dots, r ,\ i = 1, \dots, m ,\ (\xi; x, y) \in \Delta_a .$$

Note that each component $g_{ik}(\xi; x, y)$ of the fixed element $g = T_z g$ is certainly an absolute continuous function in ξ for every (x, y), is Lipschitzian in y of constant $1 + p$ for every (ξ, x), and satisfies

$$|\partial g_{ik}(\xi; x, y)/\partial\xi| \leqslant m(\xi) ,$$
$$(\xi; x, y) \in I_a \times I_a \times E^r , \qquad \text{(a.e.)},\ i = 1, \dots, m ,\ k = 1, \dots, r .$$

Moreover, for every $i = 1, \dots, m$, the r-vector function $\breve{g}_i(\xi; x, y) = (g_{ik},\ k = 1, \dots, r)$, thought of as a function of ξ, is a Carathéodory solution of the system of ordinary differential equations

$$(2.25) \qquad \mathrm{d}g_{ik}(\xi; x, y)/\mathrm{d}\xi = \varrho_{ik}(\xi, \breve{g}_i(\xi; x, y), z(\xi, \breve{g}_i(\xi; x, y))) ,\ 0 \leqslant \xi \leqslant a ,\ \text{(a.e.)} ,$$

$$(2.26) \qquad g_{ik}(x; x, y) = y_k , \qquad k = 1, \dots, r .$$

Let us prove that each component $g_{ik}(\xi; x, y)$ of the fixed element $g[z]$ is absolutely continuous in x for every (ξ, y). Indeed, for any two $(\xi; x, y)$, $(\xi; \bar{x}, y) \in \Delta_a$, we have

$$(2.27) \quad \left\{ \begin{aligned} & |g_{ik}(\xi; x, y) - g_{ik}(\xi; \bar{x}, y)| = \\ & = \Big| \int_\xi^x \varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y)))\, \mathrm{d}\alpha - \\ & - \int_\xi^{\bar{x}} \varrho_{ik}(\alpha, \breve{g}_i(\alpha; \bar{x}, y), z(\alpha, \breve{g}_i(\alpha; \bar{x}, y)))\, \mathrm{d}\alpha \Big| \leqslant \\ & \leqslant \Big| \int_x^{\bar{x}} m(\alpha)\, \mathrm{d}\alpha \Big| + \Big| \int_\xi^x l(\alpha)(1+Q)|\breve{g}_i(\alpha; x, y) - \breve{g}_i(\alpha; \bar{x}, y)|\, \mathrm{d}\alpha . \end{aligned} \right.$$

Since for every x, $\bar{x}$, y and i fixed,

$$\delta = \max_k \max\left[\,|\breve{g}_i(\xi;\, x, y) - \breve{g}_i(\xi;\, \bar{x}, y)|,\; 0 \leqslant \xi \leqslant a\right],$$

is certainly attained for some k and some $\bar{\xi}$ (δ depends on x, $\bar{x}$, y, i), we derive from (2.27) that

$$\delta \leqslant \Big|\int_x^{\bar{x}} m(\alpha)\, \mathrm{d}\alpha\Big| + L_a(1+Q)\,\delta\,,$$

or

$$|g_{ik}(\xi;\, x, y) - g_{ik}(\xi;\, \bar{x}, y)| \leqslant (1 - L_a(1+Q))^{-1}\Big|\int_x^{\bar{x}} m(\alpha)\, \mathrm{d}\alpha\Big|\,,$$

$$0 \leqslant \xi \leqslant a\,, \qquad k = 1, \ldots, r\,.$$

This proves that each $g_{ik}[z](\xi;\, x, y)$ is an absolutely continuous function of x for every (ξ, y) with

$$|\partial g_{ik}[z](\xi;\, x, y)/\partial x| \leqslant (1 - L_a(1+Q))^{-1} m(x) = \lambda m(x)\,,$$

(a.e.), $i = 1, \ldots, m$, $k = 1, \ldots, r$.

Because of (2.4) and (2.19) we know that $\breve{g}_i(\xi;\, x, y)$ is the unique solution of problem (2.25), (2.26). Thus, $\breve{g}_i$ satisfies the groupal property

$$\breve{g}_i(\xi';\, \xi, \breve{g}_i(\xi;\, x, y)) = \breve{g}_i(\xi';\, x, y)\,, \qquad 0 \leqslant \xi\,,\ \xi' \leqslant a.$$

For ξ', ξ, x, y replaced by ξ, x, 0, η, or in particular by 0, x, 0, η, we have

$$\breve{g}_i(\xi;\, x, \breve{g}_i(x;\, 0, \eta)) = \breve{g}_i(\xi;\, 0, \eta)\,, \tag{2.28}$$

$$\breve{g}_i(0;\, x, \breve{g}_i(x;\, 0, \eta)) = \breve{g}_i(0;\, 0, \eta) = \eta\,. \tag{2.29}$$

Thus, for $y = \breve{g}_i(x;\, 0, \eta)$, the symmetric relations hold

$$y = \breve{g}_i(x;\, 0, \eta)\,, \qquad \eta = \breve{g}_i(0;\, x, y)\,. \tag{2.30}$$

For any fixed $z \in \mathscr{K}_1$ and $x \in I_a$, these relations represent a $1-1$ transformation of the y-space E^r into the η-space E^r. Indeed, if $y_1 = \breve{g}_i(x;\, 0, \eta_1) = \eta_1 + \breve{h}_i(x;\, 0, \eta_1)$, $y_2 = \breve{g}_i(x;\, 0, \eta_2) = \eta_2 + \breve{h}_i(x;\, 0, \eta_2)$, then

$$|y_1 - y_2| = |\eta_1 - \eta_2 + (\breve{h}_i(x;\, 0, \eta_1) - \breve{h}_i(x;\, 0, \eta_2)|\,,$$

and hence

$$(1-p)\,|\eta_1-\eta_2|\leqslant|y_1-y_2|\leqslant(1+p)\,|\eta_1-\eta_2|\,,$$

where $0<p<1$. Analogously, we could prove that

$$(1-p)\,|y_1-y_2|\leqslant|\eta_1-\eta_2|\leqslant(1+p)\,|y_1-y_2|\,.$$

By adding equation $x=x$ to relations (2.30), we obtain a $1-1$ transformation of the slab $I_a\times E^r$ of the xy-space E^{r+1} onto the slab $I_a\times E^r$ of the $x\eta$-space E^{r+1}.

Finally, we consider the operation $z\to g[z]$, or $\mathscr{K}_1\to\mathscr{K}_c$, mapping each element $z\in\mathscr{K}_1$ into the corresponding element $g=g[z]\in\mathscr{K}_0$. By taking as usual $\breve{g}_i=y+\breve{h}_i$, we have a transformation $z\to h[z]$, or $\mathscr{K}_1\to\mathscr{K}_0$, mapping each element $z\in\mathscr{K}_1$ into the fixed point $h=\tilde{T}_z h$, or $h[z]$, of the transformation $\tilde{T}_z$. Let us prove that $z\to h[z]$ is a continuous map.

To this effect, let $z,\,z'\in\mathscr{K}_1$, and let us denote by $h,\,h'$ the corresponding elements in $\mathscr{K}_0$, or fixed points $h=\tilde{T}_{z'}h$, $h'=\tilde{T}_z h'$. From (2.24) we derive now

$$|h_{ik}(\xi;\,x,y)-h'_{ik}(\xi;\,x,y)|=|\int_\xi^x[\varrho_{ik}(\alpha,\breve{g}_i(\alpha;\,x,y),z(\alpha,\breve{g}_i(\alpha;\,x,y)))-$$

$$-\varrho_{ik}(\alpha,\breve{g}'_i(\alpha;\,x,y),z'(\alpha,\breve{g}'_i(\alpha;\,x,y)))]\,\mathrm{d}\alpha|\leqslant|\int_\xi^x l(\alpha)((1+Q)\|h-h'\|+\|z-z'\|)\,\mathrm{d}\xi\,.$$

Hence,

$$\|h-h'\|\leqslant L_a(1+Q)\|h-h'\|+L_a\|z-z'\|\,,$$

where $L_a(1+Q)<1$, and this yields

$$\|h-h'\|\leqslant(1-L_a(1+Q))^{-1}L_a\|z-z'\|=\lambda L_a\|z-z'\|\,.$$

It is correct to write this relation in the form

$$\|g-g'\|\leqslant\lambda L_a\|z-z'\|\,. \tag{2.31}$$

(d) *The transformation* T^*_φ: Here z denotes any element of $\mathscr{K}_1$ and $g=g[z]\in\mathscr{K}_0$ the unique fixed element $g=T_z g\in\mathscr{K}_0$ of the transformation T_z.

Let $\mathscr{I}$ denote the class of all functions $\varphi(y)=(\varphi_1,\dots,\varphi_m)$, $y\in E^r$, such that, for all $y,\,\bar{y}\in E^r$ and $i=1,\dots,m$, we have

$$|\varphi_i(y)|\leqslant\omega\,,\qquad|\varphi_i(y)-\varphi_i(\bar{y})|\leqslant\Lambda\,|y-\bar{y}|\,. \tag{2.32}$$

For every $\varphi \in \mathscr{I}$ let us consider the set $\mathscr{K}_{1\varphi}$ of all systems $z(x, y) = (z_1, \ldots, z_m)$ of the class $\mathscr{K}_1$ satisfying the conditions

$$z_i(0, y) = \varphi_i(y), \qquad y \in E^r, \; i = 1, .., m. \tag{2.33}$$

The class $\mathscr{K}_{1\varphi}$ is not empty. Indeed, the function $z(x, y) = (\varphi_1(y), \ldots, \varphi_m(y))$. $0 \leqslant x \leqslant a_0$, $y \in E^r$, is of class $\mathscr{K}_1$ since $\omega < \Omega$, $\Lambda < Q$, and z is of class $\mathscr{K}_{1\varphi}$.

Let us consider the transformation T^*_φ, or $z \to Z$, defined by

$$Z_i(x, y) = \varphi_i(\breve{g}_i(0; x, y)) + \int_0^x f_i(\beta, \breve{g}_i(\beta; x, y), z(\beta, \breve{g}_i(\beta; x, y)))\, d\beta, \tag{2.34}$$

$$(x, y) \in D_a, \qquad i = 1, \ldots, m,$$

where $g = g[z]$ is the element $g = T_z g \in \mathscr{K}_0$, or $g = [g_{ik}, i = 1, \ldots, m, k = 1, \ldots, r]$, we have determined in part (c), and $\breve{g}_i(\xi; x, y) = [g_{ik}, k = 1, \ldots, r]$, $(\xi, x, y) \in I_a \times I_a \times E^r$.

We have now, by relations (2.16),

$$Z_i(0, y) = \varphi_i(\breve{g}_i(0; 0, y)) = \varphi_i(y), \qquad y \in E^r, \; i = 1, \ldots, m,$$

and thus the functions Z_i satisfy relations (2.33). Also, because of (2.3), (2.8), (2.32), we have

$$|Z_i(x, y)| \leqslant \omega + \int_0^x h(\beta)\, d\beta \leqslant \omega + H_a \leqslant \Omega.$$

and thus the functions Z_i satisfy relations (2.18). Finally, by force of (2.5), (2.9), (2.16), (2.19), (2.32), (2.34) we have

$$|Z_i(x, y) - Z_i(x, \bar{y})| \leqslant |\varphi_i(\breve{g}_i(0; x, y)) - \varphi_i(\breve{g}_i(0; x, \bar{y}))| +$$

$$+ \int_0^x |f_i(\beta, \breve{g}_i(\beta; x, y), z(\beta, \breve{g}_i(\beta; x, y))) - f_i(\beta, \breve{g}_i(\beta; x, \bar{y}), z(\beta, \breve{g}_i(\beta; x, \bar{y})))|\, d\beta \leqslant$$

$$\leqslant \Lambda |\breve{g}_i(0; x, y) - \breve{g}_i(0; x, \bar{y})| + \int_0^x l(\beta)[|\breve{g}_i(\beta; x, y) - \breve{g}_i(\beta; x, \bar{y})| +$$

$$+ |z(\beta, \breve{g}_i(\beta; x, y)) - z(\beta, \breve{g}_i(\beta; x, \bar{y}))|]\, d\beta \leqslant$$

$$\leqslant (1 + p)(\Lambda + L_a(1 + Q))\, |y - \bar{y}| \leqslant Q\, |y - \bar{y}|$$

for all (x, y), $(x, \bar{y}) \in D_a$, $i = 1, \ldots, m$. Thus, the functions Z_i satisfy also relations (2.19), and we have proved that T^*_φ maps $\mathscr{K}_{1\varphi}$ into $\mathscr{K}_{1\varphi}$.

Let us prove that T_φ^* is a contraction. Let $z, z' \in \mathscr{K}_{1\varphi}$ and $Z = T_\varphi^* z$, $Z' = T_\varphi^* z'$. With obvious notations we have

$$|Z_i - Z_i'| = |\varphi_i(\breve{g}_i(0; x, y)) - \varphi_i(\breve{g}_i'(0; x, y))| +$$
$$+ \int_0^x |f_i(\beta, \breve{g}_i(\beta; x, y), z(\beta, \breve{g}_i(\beta; x, y))) - f_i(\beta, \breve{g}_i'(\beta; x, y), z'(\beta, \breve{g}_i'(\beta; x, y)))| \, \mathrm{d}\beta \leqslant$$
$$\leqslant \Lambda \|\breve{g}_i - \breve{g}_i'\| + \int_0^x l(\beta)((1+Q)\|\breve{g}_i - \breve{g}_i'\| + \|z - z'\|) \, \mathrm{d}\beta \leqslant$$
$$\leqslant (\Lambda + L_a(1+Q))\|\breve{g}_i - \breve{g}_i'\| + L_a\|z - z'\| .$$

Using (2.10), (2.31) we have, for all $z, z' \in \mathscr{K}_{1\varphi}$,

$$\|Z - Z'\| = \|T_\varphi^* z - T_\varphi^* z'\| \leqslant L_a(1 + \lambda(\Lambda + L_a(1+Q)))\|z - z'\| ,$$

where $\nu = L_a(1 + \lambda(\Lambda + L_a(1+Q))) < 1$.

We have proved that $T_\varphi^*: \mathscr{K}_{1\varphi} \to \mathscr{K}_{1\varphi}$ is a contraction. By BANACH's fixed point theorem, there is an element $z \in \mathscr{K}_{1\varphi}$ with $z = T_\varphi^* z$, and we denote this element by $z = z[\varphi]$. For $z = z[\varphi]$, and corresponding element $g = g[z] \in \mathscr{K}_0$ the following integral equations hold:

$$(2.35) \quad \begin{cases} g_{ik}(\xi; x, y) = y_k - \int_\xi^x \varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y))) \, \mathrm{d}\alpha , \\ \qquad (\xi, x, y) \in I_a \times I_a \times E^r , \; k = 1, \ldots, r , \; i = 1, \ldots, m , \\ z_i(x, y) \qquad = \varphi_i(\breve{g}_i(0; x, y)) + \int_\xi^x f_i(\beta, \breve{g}_i(\beta; x, y), z(\beta; \breve{g}_i(\beta; x, y))) \, \mathrm{d}\beta , \\ \qquad (x, y) \in I_a \times E^r , \; i = 1, \ldots, m , \\ z_i(0, y) \qquad = \varphi_i(y) , \qquad\qquad y \in E^r , \; i = 1, \ldots, m . \end{cases}$$

Let us prove that, for the fixed element $z = T_\varphi^* z \in \mathscr{K}_{1\varphi}$, and every $y \in E^r$, the functions $z_i(x, y)$ are absolutely continuous in x. Indeed, for all (x, y), $(\bar{x}, y) \in I_a \times E^r = D_a$ and $i = 1, \ldots, m$, we have

$$|z_i(x, y) - z_i(\bar{x}, y)| \leqslant |\varphi_i(\breve{g}_i(0; x, y)) - \varphi_i(\breve{g}_i(0; \bar{x}, y))| + |\int_x^{\bar{x}} h(\beta) \, \mathrm{d}\beta| +$$
$$+ \int_0^x |f_i(\beta, \breve{g}_i(\beta; x, y), z(\beta; \breve{g}_i(\beta; x, y))) - f_i(\beta, \breve{g}_i(\beta; \bar{x}, y), z(\beta; \breve{g}_i(\beta; \bar{x}, y)))| \, \mathrm{d}\beta \leqslant$$
$$\leqslant \Lambda(1 - L_a(1+Q))^{-1} |\int_x^{\bar{x}} m(\alpha) \, \mathrm{d}\alpha| + |\int_x^{\bar{x}} h(\alpha) \, \mathrm{d}\alpha| =$$
$$= |\int_x^{\bar{x}} [(\Lambda + L_a(1+Q)) \lambda m(\alpha) + h(\alpha)] \, \mathrm{d}\alpha| .$$

This proves that $z_i(x, y)$ is absolutely continuous in x with

$$|\partial z_i(x, y)/\partial x| \leqslant (\Lambda + L_a(1+Q))\,\lambda m(x) + h(x)\,, \qquad 0 \leqslant x \leqslant a \text{ (a.e.)}\,.$$

By taking $y = \breve{g}_i(x; 0, \eta)$ in (2.35) we have, by (2.28),

$$\text{(2.36)} \qquad \begin{aligned} z_i(x, \breve{g}_i(x; 0, \eta)) &= \varphi_i(\eta) + \int_0^x f_i(\beta, \breve{g}_i(\beta; 0, \eta), z(\beta, \breve{g}_i(\beta; 0, \eta)))\, d\beta\,, \\ z_i(0, \eta) &= \varphi_i(\eta)\,. \end{aligned}$$

Note that, for every $\eta \in E^r$ the first member of (2.36) is absolutely continuous in x as the superposition of Lipschitzian and absolutely continuous functions. From (4.ii) we derive that the relations

$$dz_i(x, \breve{g}_i(x; 0, \eta))/dx = \partial z_i/\partial x + \textstyle\sum_k (\partial g_{ik}/\partial x)(\partial z_i/\partial y_k)\,, \qquad i = 1, \ldots, m\,,$$

hold a.e. in D_a. By differentiation of (2.36), and using relations (2.25) and (2.35), we derive

$$\begin{aligned} &\partial z_i(x, \breve{g}_i(x; 0, \eta))/\partial x + \\ &\quad + \textstyle\sum_k \varrho_{ik}(x, \breve{g}_i(x; 0, \eta), z(x, \breve{g}_i(x; 0, \eta)))[\partial z_i(x, \breve{g}_i(x; 0, \eta))/\partial y_k] = \\ &\quad = f_i(x, \breve{g}_i(x; 0, \eta), z(x, \breve{g}_i(x; 0, \eta)))\,, \qquad 0 \leqslant x \leqslant a\,,\ i = 1, \ldots, m\,. \end{aligned}$$

In other words, these relations hold a.e. in the slab D_a of the $x\eta$-space. By taking $y = \breve{g}_i(x; 0, \eta)$, and using the properties of the transformation (2.30), we obtain

$$\text{(2.37)} \qquad \begin{aligned} \partial z_i(x, y)/\partial x + \textstyle\sum_k \varrho_{ik}(x, y, z(x, y))\, \partial z_i(x, y)/\partial y_k = f_i(x, y, z(x, y))\,, \\ (x, y) \in D_a\,, \qquad i = 1, \ldots, m\,, \end{aligned}$$

and these relations hold a.e. in the slab D_a of the xy-space. The existence of a unique element $z \in \mathscr{K}_1$ for every $\varphi \in \mathscr{I}$, satisfying (2.1), (2.2) as stated in Theorem I, is thereby proved.

(e) *The solution z depends continuously on the initial data φ.* For every $\varphi \in \mathscr{I}$ we have determined an element $z = z[\varphi] \in \mathscr{K}_{1\varphi}$. Let us prove that the map $\varphi \to z[\varphi]$, or $\mathscr{I} \to \mathscr{K}_1$, so defined is continuous. Indeed, if $\varphi, \varphi' \in \mathscr{I}$, if z, z' are the corresponding elements, $z = T^*_\varphi z$, $z' = T^*_{\varphi'} z'$, and $g = g[z]$, $g' =$

$= g[z']$ the corresponding elements in $\mathscr{K}_0$, then we have, by using (2.5), (2.6), (2.10), (2.19), (2.31), (2.35),

$$\begin{aligned}|z_i(x, y) - z'_i(x, y)| &= |\varphi_i(\breve{g}_i(0; x, y) - \varphi'_i(\breve{g}'_i(0; x, y)) + \\ &+ \int_0^x [f_i(\beta, \breve{g}_i(\beta; x, y), z(\beta; \breve{g}_i(\beta; x, y))) - \\ &- f_i(\beta, \breve{g}'_i(\beta; x, y), z'(\beta, \breve{g}'_i(\beta; x, y)))] \, d\beta| \leqslant \\ &\leqslant \|\varphi - \varphi'\| + \Lambda \|g - g'\| + \int_0^a l(\beta)[(1 + Q)\|g - g'\| + \|z - z'\|] \, d\beta \leqslant \\ &\leqslant \|\varphi - \varphi'\| + (\Lambda + L_a(1 + Q))\|g - g'\| + L_a\|z - z'\| \leqslant \\ &\leqslant \|\varphi - \varphi'\| + L_a(1 + \lambda(\Lambda + L_a(1 + Q)))\|z - z'\| ,\end{aligned}$$

where $\nu = L_a(1 + \lambda(\Lambda + L_a(1 + Q))) < 1$. Thus,

$$\|z - z'\| \leqslant [1 - L_a(1 + \lambda(\Lambda + L_a(1 + Q)))]^{-1}\|\varphi - \varphi'\| ,$$

or

$$\|z - z'\| \leqslant (1 - \varepsilon)^{-1}\|\varphi - \varphi'\| . \tag{2.38}$$

Theorem I is thereby proved.

Remark 1. The vector functions $\breve{g}_i(\xi; x, y)$, $0 \leqslant \xi \leqslant a$, $i = 1, \ldots, m$, with $\breve{g}_i(x; x, y) = y$, are the m real characteristic lines through the point $(x, y) \in D_a$ corresponding to the solution $z(x, y) = (z_1, \ldots, z_m)$.

Remark 2. If the function $\varphi(y) = (\varphi_1, \ldots, \varphi_m)$ is given only in an interval $B = \prod_{k=1}^{r} [b_{1k}, b_{2k}] \subset E^r$, $b_{1k} < b_{2k}$ finite, and φ satisfies (2.6) in B, then we can well extend φ to all of E^r in such a way to satisfy (2.6) in all of E^r, and then we can determine a number $a > 0$ and the solution $z(x, y) = (z_1, \ldots, z_m)$ in the infinite slab D_a together with the corresponding characteristic lines g_i. We can even arrange that φ has compact support in E^r and then z too will have compact support in D_a. Obviously z and g depend on the chosen extension.

Nevertheless, z and g are uniquely determined by the data φ on B in the finite domain R defined by

$$0 \leqslant x \leqslant \bar{a}\,, \qquad b_{1k} + M(x) \leqslant y_k \leqslant b_{2k} - M(x)\,, \qquad k = 1, \ldots, r\,,$$

where $\bar{a}$ is the maximum of the numbers $\bar{a} \leqslant a$ with $2M(\bar{a}) \leqslant b_{2k} - b_{1k}$, $k = 1, \ldots, r$, and $M(x) = \int_0^x m(\alpha)\,\mathrm{d}\alpha$, $0 \leqslant x \leqslant a_0$.

Remark 3. Note that, if the functions $\varrho_{ik}(x, y, z)$, $f_i(x, y, z)$ in Theorem I are defined in $D_{a_0} \times E^m$ and satisfy there all relations required in I, then there is no need to require (2.7), and requirement $\omega + H_a \leqslant \Omega$ in (2.8) can be dropped. Indeed, then in the proof of I we need not require that the functions z_i in the class $\mathscr{K}_1$ are bounded by Ω, and we need not require (2.18). Yet the corresponding functions z_i are bounded in D_a, namely $|z_i| \leqslant \omega + H_a$. Thus, requirement (2.7) in I can be dropped if the functions ϱ_{ik}, f_i are defined in $D_a \times E^m$ and satisfy there all other requirements of I.

Remark 4. We have pointed out in the proof of (2.37), that the first members of such equations are, a.e. in D_a, the total derivatives $\mathrm{d}z(x, \breve{g}_i(x;\, 0, \eta))/\mathrm{d}x$. There may well be a set of measure zero of points $\eta \in E^r$ for which $\mathrm{d}z(x, \breve{g}_i(x;\, 0, \eta))/\mathrm{d}x$ exists for almost all x, but the first members of (2.37) do not. This can be seen by the following example with $r = m = 1$, $\partial z/\partial x + \partial z/\partial y = f(x, y)$, $x \geqslant 0$, $-\infty < y < +\infty$, with $f(x, y) = y$ for $x \leqslant y$ and $f(x, y) = x$ for $y \leqslant x$, and $z(0, y) = 0$. Its solution is $z(x, y) = xy - 2^{-1}x^2$ for $y \geqslant x$, $z(x, y) = 2^{-1}x^2$ for $y \leqslant x$, as one can easily verify. On the half straight line $y = x > 0$, $z(x, x) = 2^{-1}x^2$ has derivative $\mathrm{d}z/\mathrm{d}x = x$, while $\partial z/\partial x$ and $\partial z/\partial y$ do not exist.

Remark 5. If the functions $\varphi_i(y)$, $y \in E^r$, are known to be periodic of some periods $T_1, \ldots, T_r$ in $y_1, \ldots, y_r$, and all functions $\varrho_{ik}(x, y, z)$, $f_i(x, y, z)$ are also periodic of the same periods in $y_1, \ldots, y_r$, then, under the same hypotheses of I, the solution $z(x, y) = (z_1, \ldots, z_m)$ is also periodic of the same periods in $y_1, \ldots, y_r$. To see this one has only to repeat the proof of I taking classes $\mathscr{K}_0$ and $\mathscr{K}_1$ made up of functions $g(\xi;\, x, y)$ and $z(x, y)$ satisfying all requirements (2.12), (2.13), (2.14), (2.18), (2.19) and in addition periodic with respect to $y_1, \ldots, y_r$ of periods $T_1, \ldots, T_r$.

3. - The existence theorem for the boundary value problem.

We consider here hyperbolic systems of the same canonic form (1.1), or

$$\partial z_i/\partial x + \sum_{k=1}^{r} \varrho_{ik}(x, y, z)\, \partial z_i/\partial y_k = f_i(x, y, z)\,, \qquad (x, y) \in D_a\,,\ i = 1, \ldots, m\,, \tag{3.1}$$

with boundary conditions

$$\sum_{j=1}^{m} b_{ij}(y)\, z_j(a_i, y) = \psi_i(y)\,, \qquad y \in E^r\,,\ i = 1, \ldots, m\,, \tag{3.2}$$

where $\psi_i(y)$, $b_{ij}(y)$ are given functions of y in E^r with $\det[b_{ij}] \neq 0$, and where $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_m \leqslant a$ are given numbers (between 0 and a). As mentioned in the Introduction, we assume that the $m \times m$ matrix $[b_{ij}]$ has « dominant » diagonal terms, and, to simplify notations, we simply assume that

$$b_{ij}(y) = \delta_{ij} + \tilde{b}_{ij}(y)\,, \qquad i, j = 1, \ldots, m\,,\ y \in E^r\,,$$

with $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$, and that

$$\sum_{j=1}^{m} |\tilde{b}_{ij}(y)| \leqslant \sigma < 1\,, \qquad i = 1, \ldots, m\,, \tag{3.3}$$

where σ is a fixed number, $0 \leqslant \sigma < 1$. Then, system (3.2) becomes

$$z_i(a_i, y) = \psi_i(y) - \sum_{j=1}^{m} \tilde{b}_{ij}(y)\, z_j(a_i, y)\,, \qquad y \in E^r,\ i = 1, \ldots, m\,.$$

In the existence theorem below we state conditions in terms of the elements ϱ_{ik}, f_i, b_{ij}, ψ_i under which problem (3.1), (3.2) has a solution, and this solution is unique in a suitable class.

Theorem II. (*An existence theorem for the boundary value problem*). *Let $\Omega > 0$, $m(x)$, $h(x)$, $l(x)$, $0 \leqslant x \leqslant a_0$, be as in Theorem I, let conditions (2.3), (2.5) of Theorem I be satisfied, and let ω_0, σ, τ, Λ_0 be constants such that*

$$0 < \omega_0 < \Omega\,, \quad 0 \leqslant \sigma < 1\,, \quad \tau \geqslant 0\,, \quad \Lambda_0 \geqslant 0\,, \quad \omega_0 < (1 - \sigma)\Omega\,. \tag{3.4}$$

Let $\psi_i(y)$, $b_{ij}(y) = \delta_{ij} + \tilde{b}_{ij}(y)$, $y \in E^r$, $i, j = 1, \ldots, m$, be functions defined in E^r such that, for all $y, \bar{y} \in E^r$ and $i = 1, \ldots, m$, we have

$$|\psi_i(y)| \leqslant \omega_0\,, \qquad |\psi_i(y) - \psi_i(\bar{y})| \leqslant \Lambda_0 |y - \bar{y}|\,, \tag{3.5}$$

$$\textstyle\sum_j |\tilde{b}_{ij}(y)| \leqslant \sigma\,, \qquad \sum_j |b_{ij}(y) - b_{ij}(\bar{y})| \leqslant \tau |y - \bar{y}|\,. \tag{3.6}$$

Then, there is a number a, $0 < a \leqslant a_0$, and continuous functions $z(x, y) = (z_1, \ldots, z_m)$, $(x, y) \in D_a$, absolutely continuous in x for every y, uniformly Lip-

schitzian in y *for every* x, *satisfying* $-\Omega \leqslant z_i(x, y) \leqslant \Omega$ *for all* $(x, y) \in D_a$, $i = 1, \ldots, m$, *satisfying* (3.1) *a.e. in* D_a *and* (3.2) *everywhere in* E^r.

This element z is unique (in the class which will be stated in the proof) and depends continuously on ψ.

Proof. First, let us choose constants ω, Λ such that

$$0 < \omega_0 < \omega < \Omega\,, \qquad \omega_0 < (1-\sigma)\omega\,,$$

$$0 \leqslant \Lambda_0 < \Lambda\,, \qquad \Lambda_0 + \tau\omega < (1-\sigma)\Lambda\,.$$

We may write the last relation in the form $\Lambda_0 + \tau\omega + \sigma\Lambda < \Lambda$. Then, let us choose a number p, $0 < p < 1$, sufficiently small so that

$$\Lambda_0 + \tau\omega + \sigma\Lambda(1+p) < (1+p)^{-1}\Lambda\,, \tag{3.7}$$

and let us take a number k, $0 < k < 1$ so that $\sigma < k < 1$. We are now in a position to apply Theorem I in connection to the constants ω and Λ just now determined. As in Section 2, part (a) of the proof of Theorem I, we use the numbers p, k already chosen, and we choose a constant Q such that $Q > (1+p)\Lambda$. Also, as indicated there, for any a, $0 < a \leqslant a_0$, sufficiently small we denote by λ and ε the constants

$$\lambda = (1 - L_a(1+Q))^{-1} > 1\,, \qquad \varepsilon = L_a(1 + \lambda(\Lambda + L_a(1+Q))) < 1\,. \tag{3.8}$$

We now choose a, $0 < a \leqslant a_0$, sufficiently small so that besides the requirements stated in Section 2, in particular relations (2.8), (2.10), we also have

$$\begin{cases} \omega_0 + \sigma\omega + (\sigma+1)H_a \leqslant \omega\,, \\ \Lambda_0 + \tau(\omega + H_a) + \sigma\Lambda(1+p) + (\sigma+1)L_a(1+Q)(1+p) \leqslant (1+p)^{-1}\Lambda\,, \\ k' = \sigma + (\sigma+1)\,\varepsilon(1-\varepsilon)^{-1} < 1\,. \end{cases} \tag{3.9}$$

where k' is defined by the last expression.

As in Section 2, parts (b) and (d), we define the classes $\mathscr{K}_0$, $\mathscr{K}_1$ and $\mathscr{I}$ in connection with the constants chosen above.

In Section 2, part (d), for every $\varphi \in \mathscr{I}$ we have determined a unique element $z \in \mathscr{K}_{1\varphi}$ satisfying (2.1) and (2.2). We consider now the transformation

T^{**}, or $\Phi = T^{**}\varphi$, $\varphi \in \mathscr{I}$, or $\varphi \to \Phi$, $\varphi(y) = (\varphi_1, \ldots, \varphi_m)$, $\Phi(y) = (\Phi_1, \ldots, \Phi_m)$, defined by

$$(3.10) \qquad \Phi_i(\eta) = [\Phi_i(g_i(0; a_i, y))]_{y=\breve{g}_i(a_i, 0, \eta)}, \qquad \eta \in E^r, \ i = 1, \ldots, m,$$

$$(3.11) \qquad \Phi_i(\breve{g}_i(0; a_i, y)) = \psi_i(y) - \sum_{j=1}^{m} \tilde{b}_{ij}(y) z_j(a_i, y) - \\ - \int_0^{a_i} f_i(\beta, \breve{g}_i(\beta; a_i, y), z(\beta, \breve{g}_i(\beta; a_i, y)))\, d\beta, \qquad y \in E^r, \ i = 1, \ldots, m.$$

Here $z(x, y) = (z_1, \ldots, z_m)$, or $z = z[\varphi] \in \mathscr{K}_{1\varphi} \subset \mathscr{K}_1$, is the solution of the CAUCHY problem for (3.1) with $z_i(0, y) = \varphi_i(y)$, $y \in E^r$, $i = 1, \ldots, m$, and $g(\xi; x, y) = [g_{ik}, k = 1, \ldots, r, i = 1, \ldots, m]$, or $g = g[x] \in \mathscr{K}_0$, is the system g relative to z. Thus, $\breve{g}_i(\xi; x, y) = [g_{ik}, k = 1, \ldots, r]$, $i = 1, \ldots, m$, and

$$(3.12) \qquad \begin{cases} g_{ik}(\xi; x, y) = y_k - \int_\xi^x \varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y)))\, d\alpha, \\ \qquad (\xi, x, y) \in I_a \times I_a \times E^r, \ k = 1, \ldots, r, \ i = 1, \ldots, m, \\ z_i(x, y) \quad = \varphi_i(\breve{g}_i(0; x, y)) + \int_0^x f_i(\beta, \breve{g}_i(\beta; x, y), z(\beta, \breve{g}_i(\beta; x, y)))\, d\beta, \\ \qquad (x, y) \in D_a = I_a \times E^r, \ i = 1, \ldots, m, \\ z_i(0, y) \quad = \varphi_i(y), \qquad y \in E^r, \ i = 1, \ldots, m. \end{cases}$$

First, we have $|z_i(x, y)| \leqslant \omega + H_a$, $(x, y) \in D_a$, $i = 1, \ldots, m$, and hence

$$|\Phi_i(\breve{g}_i(0; a_i, y))| \leqslant |\psi_i(y)| + |\sum_{j=1}^{m} \tilde{b}_{ij}(y) z_j(a_i, y) + \int_0^{a_i} h(\beta)\, d\beta \\ \leqslant \omega_0 + \sigma(\omega + H_a) + H_a \leqslant \omega.$$

Consequently, we have also

$$(3.13) \qquad |\Phi_i(\eta)| \leqslant \omega, \qquad \eta \in E^r, \ i = 1, \ldots, m.$$

Furthermore, we have, for all $y, \bar{y} \in E^r$, $i = 1, \ldots, m$,

$$(3.14)\quad \begin{cases} |\Phi_i(\breve{g}_i(0;\, a_i, y)) - \Phi_i(\breve{g}_i(0;\, a_i, \bar{y}))| \leqslant \\ \leqslant |\psi_i(y) - \psi_i(\bar{y})| + \sum_{j=1}^{m} |\tilde{b}_{ij}(y) - \tilde{b}_{ij}(\bar{y})|\, |z_j(a_i, y)| + \\ + \sum_{j=1}^{m} |\tilde{b}_{ij}(\bar{y})| \big[|\varphi_j(\breve{g}_i(0;\, a_i, y)) - \varphi_j(\breve{g}_j(0;\, a_i, \bar{y}))| + \\ + \int_0^{a_i} |f_j(\beta, \breve{g}_j(\beta;\, a_i, y), z(\beta, \breve{g}_j(\beta;\, a_i, y))) - \\ - f_j(\beta, \breve{g}_j(\beta;\, a_i, \bar{y}), z(\beta, \breve{g}_j(\beta;\, a_i, \bar{y})))|\, d\beta \big] + \\ + \int_0^{a_i} |f_i(\beta, \breve{g}_i(\beta;\, a_i, y), z(\beta, \breve{g}_i(\beta, a_i, y))) - \\ - f_i(\beta, \breve{g}_i(\beta;\, a_i, \bar{y}), z(\beta, \breve{g}_i(\beta;\, a_i, \bar{y})))|\, d\beta \leqslant \\ \leqslant [\Lambda_0 + \tau(\omega + H_a) + \sigma\Lambda(1+p) + \sigma L_a(1+Q)(1+p) + \\ + L_a(1+Q)(1+p)]\, |y - \bar{y}| \leqslant (1+p)^{-1}\Lambda\, |y - \bar{y}|\,. \end{cases}$$

We note here that, if a function $F(y)$, $y \in E^r$, satisfies $|F(y) - F(\bar{y})| \leqslant \leqslant K|y - \bar{y}|$, $y, \bar{y} \in E^r$, then

$$(3.15)\quad |F(\breve{g}_i(a_i,\, 0, \eta)) - F(\breve{g}_i(a_i,\, 0, \bar{\eta}))| \leqslant$$
$$\leqslant K|\breve{g}_i(a_i;\, 0, \eta) - \breve{g}_i(a_i;\, 0, \bar{\eta})| \leqslant K(1+p)|\eta - \bar{\eta}|$$

for all $\eta, \bar{\eta} \in E^r$.

Now, by force of (2.30), we have

$$\Phi_i(\eta) = [\Phi_i(\breve{g}_i(0;\, a_i, y))]_{y = \breve{g}_i(a_i;\, 0, \eta)}$$

and thus, from (3.14) and (3.15) we also have

$$(3.16)\quad |\Phi_i(\eta) - \Phi_i(\bar{\eta})| \leqslant (1+p)^{-1}\Lambda \cdot (1+p)|\eta - \bar{\eta}| = \Lambda|\eta - \bar{\eta}|$$

for all $\eta, \bar{\eta} \in E^r$, $i = 1, \ldots, m$. From (3.13) and (3.16) we see that the transformation $T^{**}\colon \varphi \to \Phi$ defined by (3.10), (3.11) maps $\mathscr{I}$ into $\mathscr{I}$.

Let us prove that $T^{*}\colon \mathscr{I} \to \mathscr{I}$ is a contraction. Indeed, if φ, φ' are elements of $\mathscr{I}$, if z, z' are the corresponding elements $z \in \mathscr{K}_{1\varphi}$, $z' \in \mathscr{K}_{1\varphi'}$, and

$g = g[z]$, $g' = g[z']$, $\Phi = T^{**}\varphi$, $\Phi' = T^{**}\varphi'$, then

$$
(3.17)\quad \begin{cases}
|\Phi_i(g_i(0;\, a_i,\, y)) - \Phi'_i(g'_i(0;\, a_i,\, y))| \leqslant \\
\leqslant \sum_{j=1}^{m} |\tilde{b}_{ij}| [\,|\varphi_j(g_j(0;\, a_i,\, y) - \varphi'_j(g'_j(0;\, a_i,\, y))| + \\
+ \int_0^{a_i} |f_j(\beta, \breve{g}_j(\beta;\, a_i,\, y), z(\beta, \breve{g}_j(\beta;\, a_i,\, y))) - \\
- f_j(\beta, \breve{g}'_j(\beta;\, a_i,\, y), z'(\beta, \breve{g}'_j(\beta;\, a_i,\, y)))|\, d\beta] + \\
+ \int_0^{a_i} |f_i(\beta, \breve{g}_i(\beta;\, a_i,\, y), z(\beta, \breve{g}_i(\beta;\, a_i,\, y))) - \\
- f_i(\beta, \breve{g}'_i(\beta;\, a_i,\, y), z'(\beta, \breve{g}'_i(\beta;\, a_i,\, y)))|\, d\beta \leqslant \\
\leqslant \sigma[\|\varphi - \varphi'\| + \Lambda\|g - g'\| + L_a(1+Q)\|g - g'\| + L_a\|z - z'\|\,] + \\
+ L_a(1+Q)\|g - g'\| + L_a\|z - z'\| \,.
\end{cases}
$$

Also we have

$$
|\Phi_i(\breve{g}_i(0;\, a_i,\, y)) - \Phi'_i(\breve{g}_i(0;\, a_i,\, y))| \leqslant
$$

$$
\leqslant |\Phi_i(\breve{g}_i(0;\, a_i,\, y) - \Phi'_i(\breve{g}'_i(0;\, a_i,\, y))| + |\,|\Phi'_i(\breve{g}'_i(0;\, a_i,\, y)) - \Phi'_i(\breve{g}_i(0;\, a_i,\, y))| \leqslant
$$

$$
\leqslant |\Phi_i(\breve{g}_i(0;\, a_i,\, y) - \Phi'_i(\breve{g}'_i(0;\, a_i,\, y))| + \Lambda\|g - g'\| \,.
$$

By using (2.31), (2.38), (3.9), (3.17) we have now

$$
\|\Phi - \Phi'\| \leqslant \sigma[\|\varphi - \varphi'\| + (\Lambda + L_a(1+Q))\,\lambda L_a\|z - z'\| + L_a\|z - z'\|] +
$$

$$
+ L_a(1+Q)\,\lambda L_a\|z - z'\| + L_a\|\,z - z'\| + \Lambda\lambda L_a\|z - z'\| \leqslant
$$

$$
\leqslant \sigma\|\varphi - \varphi'\| + (\sigma + 1)L_a[1 + \lambda(\Lambda + L_a(1+Q))]\|z - z'\| =
$$

$$
= [\sigma + (\sigma+1)\,\varepsilon(1-\varepsilon)^{-1}]\|\varphi - \varphi'\| = k'\|\varphi - \varphi'\| \,,
$$

where $k' < 1$. Thus, $T^{**}: \mathscr{I} \to \mathscr{I}$ is a contraction. By BANACH's fixed point theorem there is an element $\varphi \in \mathscr{I}$ with $\varphi = T^{**}\varphi$.

For this element $\varphi = T^{**}\varphi \in \mathscr{I}$ and corresponding element $z = z[\varphi] \in \mathscr{K}_{1\varphi} \subset \subset \mathscr{K}_1$ we derive from (3.11)

$$
\varphi_i(\breve{g}_i(0;\, a_i,\, y)) = \psi_i(y) - \sum_{j=1}^{m} \tilde{b}_{ij}(y)\, z_j(a_i,\, y) - \int_0^{a_i} f_i(\beta, \breve{g}_i(\beta;\, a_i,\, y), z(\beta, g_i(\beta;\, a_i,\, y)))\, d\beta
$$

or, by (3.12), also

$$z(a_i, y) = \psi_i(y) - \sum_{j=1}^{m} \tilde{b}_{ij}(y) z_j(a_i, y) , \qquad y \in E^r , \ i = 1, \dots, m .$$

The existence of the element $z(x, y)$, $(x, y) \in D_a$ stated in Theorem II is thereby proved and this element is unique in the class $\mathscr{K}_1$ with initial data $z_i(0, y) = \varphi_i(y)$, $y \in E^r$, $i = 1, \dots, m$, $\varphi = (\varphi_1, \dots, \varphi_m) \in \mathscr{I}$.

Let us prove that this element z depends continuously on ψ. Indeed, if ψ, ψ' are any two corresponding elements, and φ, φ' the corresponding initial values, then by repeating the argument above with $\Phi = \varphi$, $\Phi' = \varphi'$, we have

$$\|\varphi - \varphi'\| \leqslant \|\psi - \psi'\| + k'\|\varphi - \varphi'\| , \qquad k' < 1 ,$$

and then, by (2.38), also

$$\|\varphi - \varphi'\| \leqslant (1 - k')^{-1} \|\psi - \psi'\| ,$$

$$\|z - z'\| \leqslant (1 - \varepsilon)^{-1} \|\varphi - \varphi'\| \leqslant (1 - \varepsilon)^{-1} (1 - k')^{-1} \|\psi - \psi'\| .$$

Theorem II is thereby proved.

4 - Some remarks on differentiation.

The statements (4.i), (4.ii) below are essentially known. For (4.ii), which was used in Section 2, a number of references are given in [**3**]. Nevertheless, we give here simple proofs for the convenience of the reader.

(a) A Lemma on L_1-convergence.

(4.i). *If $f(x)$, $f_k(x)$, $x \in D$, $f, f_k \in L_1(D)$, $k = 1, 2, \dots$, are functions defined in a domain D of E^r, and $f_k \to f$ strongly in $L_1(D)$ as $k \to \infty$, then there are a functions $\bar{f}(x) \geqslant 0$, $x \in D$, $\bar{f} \in L_1(D)$, and a sequence $[k_s]$ with $k_s \to \infty$, such that $|f_{k_s}(x)| \leqslant \bar{f}(x)$ for all s and almost all $x \in D$.*

Proof. By replacing f_k by $f_k - f$, we see that it is not restrictive to assume $f = 0$ in (4.i). Thus, $f_k \to 0$ strongly in $L_1(D)$. Hence, for every integer $s = 1, 2, \dots$, there is a first index $k_s \geqslant s$ such that $\int_D |f_{k_s}| \, dx < 2^{-s}$. Let

$$F(x) = \lim_{s \to \infty} F_s(x) , \qquad F_s(x) = \operatorname{Max} \left[|f_{k_1}(x)|, \dots, |f_{k_s}(x)| \right] .$$

Then $0 \leqslant F_s(x) \leqslant F_{s+1}(x)$, $x \in D$, and

$$\int_D F_s(x)\,\mathrm{d}x \leqslant \sum_{j=1}^{s} \int_D |f_{k_j}(x)|\,\mathrm{d}x \leqslant 2^{-1} + \ldots + 2^{-s} < 1 .$$

By B. LEVI's theorem, then $F \in L(D)$, $\int_D F(x)\mathrm{d}x \leqslant 1$, F is finite almost everywhere in D, and $|f_{k_s}(x)| \leqslant F_s(x) \leqslant F(x)$, $x \in D$. Lemma (4.i) is thereby proved.

(b) A chain rule differentiation Lemma.

(4.ii). *Let* $I = [a, b] \times \prod_1^r [a_i, b_i] \subset E^{r+1}$, *and let* $z(x, y)$, $(x, y) \in I$, $y = (y_1, \ldots, y_r)$, *be a continuous function on* I. *Let us assume that there is a constant* $C \geqslant 0$ *and a function* $\mu(x) \geqslant 0$, $x \in [a, b]$, $m \in L_1[a, b]$, *such that for all* (x, y), $(\bar{x}, y)$, $(x, \bar{y}) \in I$ *we have*

$$|z(x, y) - z(\bar{x}, y)| \leqslant \left|\int_x^{\bar{x}} \mu(\alpha)\,\mathrm{d}\alpha\right| , \qquad |z(x, y) - z(x, \bar{y})| \leqslant C|y - \bar{y}| .$$

Let D *be a domain in a* η*-space* E^s, *and let* $\varphi(x, \eta) = (\varphi_1(x, \eta), \ldots, \varphi_r(x, \eta))$ *be an* r*-vector function defined in* $[a, b] \times D$, *and let us assume that each* $\varphi_i(x, \eta)$ *is continuous in* $[a, b] \times D$, *has values in* $[a_i, b_i]$, *and there is some function* $m_i(x) \geqslant 0$, $x \in [a, b]$, $m_i \in L_1[a, b]$, *such that for all* (x, η), $(\bar{x}, \eta) \in [a, b] \times D$, $i = 1, \ldots, r$, *we have*

$$|\varphi_i(x, \eta) - \varphi_i(\bar{x}, \eta)| \leqslant \left|\int_x^{\bar{x}} m_i(\alpha)\,\mathrm{d}\alpha\right| .$$

Then, $Z(x, \eta) = z(x, \varphi(x, \eta))$, $(x, \eta) \in [a, b] \times D$, *is continuous in* $[a, b] \times D$, *is absolutely continuous in* x *for every* $\eta \in D$, *and almost everywhere in* $[a, b] \times D$ *we have*

$$\partial Z/\partial x = \partial z/\partial x + \sum_{k=1}^{r} (\partial z/\partial y_k)(\partial \varphi_k/\partial x) , \tag{4.1}$$

where the arguments of Z *are* (x, η), *the arguments in* $\partial z/\partial x$, $\partial z/\partial y_k$ *are* $(x, \varphi(x, \eta))$, *and the arguments in* $\partial \varphi_k/\partial x$ *are* (x, η).

Proof. The continuity of $Z(x, \eta)$ in $[a, b] \times D$ and its absolute continuity with respect to x for every $\eta \in D$ are straightforward. To prove (4.1) a.e. in $[a, b] \times D$, it is enough to prove it locally, nemely, in arbitrary subregions

$[\bar{\alpha}, \bar{\beta}] \times \bar{D} \subset [a, b] \times D$, say small enough so that, in each of the $r+1$ double inequalities $a \leqslant \bar{\alpha} < \bar{\beta} \leqslant b$, $a_i \leqslant \varphi_i(x, \eta) \leqslant b_i$ for $(x, \eta) \in [\bar{\alpha}, \bar{\beta}] \times \bar{D}$, only one $=$ sign at most may hold. Let us assume, for example, that $a \leqslant \bar{\alpha} < \bar{\beta} < b$, $a_i \leqslant \varphi_i(x, \eta) < b_i$, and hence also $a \leqslant \bar{\alpha} < \beta \leqslant b - h_0$, $a_i \leqslant \varphi_i(x, \eta) \leqslant b_i - h_0$ for $(x, \eta) \in [\bar{\alpha}, \bar{\beta}] \times \bar{D}$, and some fixed $h_0 > 0$. For any h, $0 < h \leqslant h_0$, let z_h denote the mean value function

$$z_h(x, y) = h^{-r-1} \int z(\xi, \lambda)\, d\xi\, d\lambda\,, \qquad (x, y) \in I_0 = [a, b - h_0] \times \prod_1^r [a_i, b_i - h_0]\,,$$

where $\lambda = (\lambda_1, \ldots, \lambda_r)$, $d\lambda = d\lambda_1 \ldots d\lambda_r$, and $\int$ ranges over all (ξ, λ) with $x \leqslant \xi \leqslant x + h$, $y_i \leqslant \lambda_i \leqslant y_i + h$, $i = 1, \ldots, r$. We know from ([**6**], p. 254 (M5), p. 256 (M15, 17); [**1**], pp. 460-468) that z_h is continuously differentiable, and

$$(4.2) \qquad \begin{cases} z_h \to z \text{ uniformly in } I_0 \text{ as } h \to 0^+\,, \\ \partial z_h/\partial x \to \partial z/\partial x\,, \qquad \partial z_h/\partial y_k \to \partial z/\partial y_k\,, \qquad k = 1, \ldots, r\,, \\ \text{strongly in } L_1[I_0] \text{ as } h \to 0^+\,. \end{cases}$$

Moreover, we have

$$(4.3) \qquad |\partial z_h/\partial x| \leqslant \mu_h(x) = h^{-1} \int_x^{x+h} \mu(\alpha)\, d\alpha\,, \qquad |\partial z_h/\partial y_k| \leqslant C\,.$$

Note that, for every $\eta \in D$, the relation

$$(4.4) \qquad \partial z_h(x, \varphi(x, \eta))/\partial x = \partial z_h/\partial x + \sum_{k=0}^r (\partial z_h/\partial y_k)(\partial \varphi_k/\partial x)$$

holds for almost all $x \in [\bar{\alpha}, \bar{\beta}]$, namely, at all those points x where the r derivatives $\partial \varphi_k/\partial x$, $k = 1, \ldots, r$, exist and are finite. Since both members of (4.4) are measurable functions, relation (4.4) holds almost everywhere in $[\bar{\alpha}, \bar{\beta}] \times \bar{D}$. Then, for any $[\alpha, \beta] \subset [\bar{\alpha}, \bar{\beta}]$, $D_0 \subset \bar{D}$, from (4.4) by integration, we obtain

$$(4.5) \qquad \int_D [z_h(\beta, \varphi(\beta, \eta)) - z_h(\alpha, \varphi(\alpha, \eta))]\, d\eta = \\ = \int_D \int_\alpha^\beta [\partial z_h/\partial x + \sum_{k=1}^r (\partial z_h/\partial y_k)(\partial \varphi_k/\partial x)]\, d\eta\, dx\,.$$

Note that $\mu_h \to \mu$ strongly in L_1 as $h \to 0^+$, again from ([**6**], p. 254). Hence, by force of (4.i), there is a sequence $[h_s]$, with $h_s > h_{s+1} > 0$, $h_s \to 0$ as $s \to \infty$, and a fixed function $\bar{\mu}(x)\geqslant 0$, $\mu\in L_1$, such that $0\leqslant\mu_{h_s}(x)\leqslant\bar{\mu}(x)$ for all s and almost all x. Thus

$$(4.6)\qquad |\partial z_{h_s}/\partial x|\leqslant\bar{\mu}(x)\,,\qquad |\partial\varphi_k/\partial x|\leqslant m_k(x)\,,\qquad |\partial z_{h_s}/\partial y_k|\leqslant C\,,$$

$$k=1,\dots,r\,,\ s=1,2,\dots\,.$$

By force of (4.2), (4.3), (4.6), and LEBESGUE dominated convergence theorem, from (4.5) we derive now

$$(4.7)\qquad \int_D [z(\beta,\varphi(\beta,\eta))-z(\alpha,\varphi(\alpha,\eta))]\,\mathrm{d}\eta=$$

$$=\int_D\int_\alpha^\beta \Big[\partial z/\partial x+\sum_{k=1}^r(\partial z/\partial y_k)(\partial\varphi_k/\partial x)\Big]\,\mathrm{d}\eta\,\mathrm{d}x\,.$$

Note that, for every $[\alpha,\beta]\subset[\bar{\alpha},\bar{\beta}]$, relation (4.7) holds for all $D\subset\bar{D}$. Hence, for every $[\alpha,\beta]\subset[\bar{\alpha},\bar{\beta}]$, the relation

$$(4.8)\qquad z(\beta,\varphi(\beta,\eta))-z(\alpha,\varphi(\alpha,\eta))=\int_\alpha^\beta\Big[\partial z/\partial x+\sum_{k=1}^r(\partial z/\partial y_k)(\partial\varphi_k/\partial x)\Big]\,\mathrm{d}x$$

holds for almost all $\eta\in\bar{D}$. Conceiving (α,β) as variables in the domain $R=[(\alpha,\beta)\,|\,\bar{\alpha}\leqslant\alpha\leqslant\beta\leqslant\bar{\beta}]$, we conclude that for every $(\alpha,\beta)\in R$, (4.8) holds for almost all $\eta\in\bar{D}$. Since both members of (4.8) are measurable in $R\times\bar{D}$, we conclude that (4.8) holds a.e. in $R\times\bar{D}$. Thus, there is a set $H_1\subset\bar{D}$, $|H_1|=0$, such that, for every $\eta\in\bar{D}-H_1$, (4.8) holds for almost all $(\alpha,\beta)\in R$. There is also a set $H_2\subset\bar{D}$, $|H_2|=0$, such that, for $\eta\in\bar{D}-H_2$, the function in the integral in (4.8) is L-integrable. Then, for $\eta\in\bar{D}-(H_1\cup H_2)$, relation (4.8) actually holds for *all* $(\alpha,\beta)\in R$. By differentiation, we see that, for all $\eta\in\bar{D}-(H_1\cup H_2)$, relation

$$(4.9)\qquad \partial z(x,\varphi(x,\eta))/\partial x=\partial z/\partial x+\sum_{k=1}^r(\partial z/\partial y_k)(\partial\varphi_k/\partial x)$$

holds for almost all $x\in[\bar{\alpha},\bar{\beta}]$. Thus, (4.9) holds a.e. in $[\bar{\alpha},\bar{\beta}]\times\bar{D}$. We have proved that (4.1) holds a.e. in $[a,b]\times D$. Statement (4.ii) is thereby proved.

Remark. We have proved (4.1) a.e. in $[a,b]\times D$. Thus, for almost all $\eta\in D$, relation (4.1) certainly holds for almost all $x\in[a,b]$. It may well occur that, for some $\eta\in D$ (at most a set of measure zero in D), relation (4.1) does not hold for almost all $x\in[a,b]$. This occurrence is exhibited by the following example. Take $r=1$, $x\geqslant 0$, $-\infty<y<+\infty$, and $z(x,y)=xy-2^{-1}x^2$ for $y\geqslant x$, $z(x,y)=2^{-1}x^2$ for $y\leqslant x$. Let $\varphi(x,\eta)=\eta+x$ for $x\geqslant 0$, $-\infty<\eta<+\infty$, and note that $Z(x, \eta) = z(x, \varphi(x, \eta)) = \eta x + 2^{-1}x^2$ for $\eta \geqslant 0$; $= 2^{-1}x^2$ for $\eta \leqslant 0$. For $\eta = 0$, we have $Z(x, 0) = 2^{-1}x^2$ and $\mathrm{d}Z(x, 0)/\mathrm{d}x = x$. On the other hand, for $y = x > 0$, neither partial derivative $\partial z/\partial x$, $\partial z/\partial y$ exists, and the second member of (4.9) is not defined.

References.

[1] L. CESARI, *Surface Area*, Univ. Press., Princeton 1955.

[2] M. CINQUINI CIBRARIO e S. CINQUINI, *Equazioni alle Derivate Parziali di Tipo Iperbolico*, Ed. Cremonese, Roma 1964.

[3] M. CINQUINI CIBRARIO, *Sistemi di equazioni alle derivate parziali in più variabili indipendenti*, Ann. Mat. Pura Appl. (4) **44** (1957), 357-417.

[4] R. COURANT and P. D. LAX, *On nonlinear partial differential equations with two independent variables*, Comm. Pure Appl. Math. **2** (1949), 255-273.

[5] R. COURANT and P. D. LAX, *Cauchy's problem for nonlinear hyperbolic differential equations in two independent variables*, Ann. Mat. Pura Appl. (4) **40** (1955), 161-166.

[6] L. M. GRAVES, *The Theory of Functions of Real Variables*, Mc Graw-Hill, New York 1946.

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