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On the existence of congruences in general algebrae. (**)

ad ANTONIO MAMBRIANI per il suo 75° compleanno

1. - Introduction.

In a previous Note [1] we have formulated within a framework of general algebra some propositions that ensure simplicity from irreducibility for a wide class of algebraic structures. In this context some results on the uniqueness of the conguences on general algebrae were stated and applicated to certain types of algebrae with lattice operations. On the other hand, the problem of the existence of congruences was not investigated.

Thus in this Note we concern ourselves with some existence properties of congruences in general algebrae, obtaining a necessary and sufficient condition for the existence of congruences (Proposition 1), which, as expected, considerabily bounds the permissible algebrae. As an application of this result, we give a similar condition in general algebrae with lattice operations and complemented principal ideals (Propositions 2 and 3) (this latter is the same class of general algebrae examinated in our preceding Note [1]) (1).

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(1) This class is of interest in a axiomatic formulation of quantum theory, which refers to a complete, atomic, orthocomplemented, weakly modular and satisfying-the-covering-law lattice.

2. - Existence of congruences in general algebrae.

As far as concerns the basic concepts that will be used in the present paper, we refer to definitions and notations which are usual in the literature on general algebrae [**2**], [**3**], [**4**]: in particular we adopt here the conventions of the above quoted note on this subject [**1**].

Thus, a set will be denoted by A, its elements by $a, b, \ldots$, an algebraic operation on A by ω, the action of ω on a n_ω-tuple $(a_1, \ldots, a_{n_\omega})$ of elements of A by $a_1 \ldots a_{n_\omega}\omega$, a set of algebraic operations on A by Ω, a general algebra by (A, Ω). Furthermore, if π is a binary relation on A writing $a\,\pi\,b$ will mean that π relates a to b (2). A congruence on A will be usually denoted by π, even in the case of the trivial and diagonal congruences.

We give now the following definition.

Definition 1. *Let (A, Ω) be a general algebra, let $\tilde{\omega}$ be a derived operation* (3), *and let $\tilde{\Omega}$ be the set of all derived operations on A. An element $\tau_0 \in \tilde{\Omega}$ will be called* elementary translation *if it is a unary operation defined by*

$$\forall x \in A\,, \qquad \tau_0 : x \to a_1 \ldots a_{i-1}\, x\, a_i \ldots a_{n_\omega - 1}\,\omega$$

where $a_1, \ldots, a_{n_\omega-1} \in A$ are $n_\omega - 1$ fixed elements and ω is an algebraic n_ω-ary operation $\in \Omega$. More generally, a unary operation $\tau : A \to A$ will be called a translation if τ is the identity, or if τ can be expressed as a product of a finite number of elementary translations.

The interest of considering translations lies in the following result (4).

Proposition A. *Let (A, Ω) be a general algebra and let π be an equivalence: then π is a congruence if and only if, for any $a, b \in A$ and for any translation τ, $a\pi b \Rightarrow a\tau\pi b\tau$.*

In the following definition we take into consideration a binary relation which will used below.

Definition 2. *Let (A, Ω) be a general algebra and let S be a subset of A. Let τ be a translation. We denote by $S\tau$ the set obtained from S by applying τ*

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(2) For the definition of subalgebra, congruence and homomorphism, see the ref. [**2**], [**3**] and [**4**].

(3) For the definition of derived operation, see particularly the references quoted in footnote (2).

(4) See ref. [**3**], p. 87.

to every element of S. Given now a pair (a, b) of elements of A, and a subset S of A, we define a binary reflexive and symmetric relation R in the following way:

$$a\, R\, b (\text{mod. } S) \Leftrightarrow a = b \qquad \textit{or} \qquad a, b \in S\tau ,$$

where τ is any translation of A.

The following proposition is known (5).

Proposition B. *Let (A, Ω) be a general algebra, let S be a subset, and let R be the above defined relation on A. Let π_R be the following transitive extension of R, defined by:*

$$\forall a, b \in A , \qquad a\, \pi_R\, b(\text{mod. S}) \Leftrightarrow \exists \text{ (finite) } n ,$$

$$\exists a_0 = a, a_1, \ldots, a_n = b :\ \forall i, i = 0, 1, \ldots, n , \qquad a_{i-1} R a_i (\text{mod. } S) .$$

Then the equivalence π_R is a congruence, and it is the least congruence which has a class containing S.

The next definition allows us to consider a particular class of subsets of a general algebra (A, Ω).

Definition 3. *Let (A, Ω) be a general algebra. A subset S of A will be called a* normal subset *of (A, Ω) whenever there exists a congruence π on (A, Ω) that admits S as a congruence class* (6).

Therefore we may enunciate as follows a necessary and sufficient condition concerning the existence of a congruence on a general algebra.

Proposition 1. *Let (A, Ω) be a general algebra and let S be a subset of A. Then S is a normal subset, if and only if, for any translation τ:*

$$(\alpha) \qquad ((\exists a \in S \,|\, a\tau \in S) \Rightarrow S\tau \subset S) \text{ (7)} .$$

Proof. We first prove necessity. Let S be a normal subset with respect to a congruence π. Let τ be any translation such that an $a \in S$ exists for

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(5) See ref. [**3**], p. 98, Exer. 3 and 4.

(6) It may be stressed that every single element $a \in A$ is a congruence class for the diagonal congruence, while the entire set A is the congruence class for the trivial congruence.

(7) We note that the proposition is trivially true in the two cases of preceding footnote (6).

which $a\tau \in S$. For every $b \in S$ we have (Proposition A) $a \pi b \Rightarrow a \tau \pi b \tau$; as $a\tau \in S$ it follows $b\tau \in S$, so $S\tau \subset S$.

We prove now sufficiency by observing that the congruence π_R of Proposition B is the least congruence which has a class containing S: as the condition (α) holds, such a class coincides with S.

Thus Proposition 1 is proved.

3. - Applications.

As a first application of Proposition 1, we shall consider the particularly simple case of a general algebra endowed with an invariant element (see Def. 4 below).

Definition 4. *An invariant element e of a general algebra (A, Ω) is an element such that $\{e\}$ is a subalgebra of (A, Ω). If a general algebra (A, Ω) has an invariant element e, we will call $\tilde{\Omega}_e$ the set of all those derived operations that preserve e. Moreover, we will call e-ideal of (A, Ω) associated to the invariant element e every subalgebra of $(A, \tilde{\Omega}_e)$* [8].

Let us now suppose that a general algebra (A, Ω), endowed with an invariant element e, satisfies $\Omega \equiv \tilde{\Omega}_e$. In this case for every e-ideal S the condition of Proposition 1 is automatically satisfied, and every e-ideal is a normal subset.

As a second application of Proposition 1, we discuss here the less immediate case of a general algebra with lattice operations. The next Proposition is an easy consequence of our Proposition 1.

Proposition 2. *Let (A, Ω) be a non-trivial general algebra with lattice operations. Let A, with respect to a set of lattice operations contained in Ω, be a lattice with a zero element O. Then an ideal S (with respect to the lattice operations) is a normal subset if and only if, for any translation τ,*

$$(\alpha') \qquad (O\tau \in S \Rightarrow S\tau \subset S)\,.$$

Proof. Let us first prove necessity. As S is an ideal, $O \in S$: moreover, S is a normal subset, so the condition (α) of Proposition 1 holds. Let $\tilde{\Omega}_L$ be the set of all derived lattice operations. Thus, for every translation $\tau \in \tilde{\Omega}_L$

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(8) From this definition it follows that, if π is a congruence on a general algebra (A, Ω) endowed with an invariant element e, the set of the elements of A which π relates to e is an e-ideal of (A, Ω) associated to e.

such that the existence of $x_0 \in S$ with the property $x_0 \tau \in S$ implies $S\tau \subset S$, we have $O\tau \subset x_0 \tau$, from $O \subset x_0$ and the isotony of τ; it follows $O\tau \in S$, as S is an ideal.

Thus $(\alpha) \Rightarrow (\alpha')$.

As concerns sufficiency, the proof is the same of the one given in Proposition 1, second part.

This concludes the proof of Proposition 2.

We give here two simple examples of non-chain lattices in order to illustrate Proposition 2 (9).

As a first example, we consider the lattice L_4 with four elements $\{O, a_1, a_2, I\}$: then the ideal $[O, a_1]$ (or $[O, a_2]$) is a normal subset, as condition (α') holds, and $[O, a_1]$ (respectively $[O, a_2]$) is a O-ideal of L_4.

As a second example we take the self-dual modular lattice L_5 with five elements $\{O, a_1, a_2, a_3, I\}$: in this lattice the ideal $[O, a_1]$ is not a normal subset, as the translation $\tau: x \to (x \cup a_2) \cap a_3$ maps O on O but a_1 on a_3 (moreover $[O, a_1]$ is not a O-ideal).

In the next Proposition 3, we go over to a further condition which ensures validity of condition (α') in Proposition 2.

Proposition 3. *Let (A, Ω) be a non-trivial general algebra with lattice operations. Let A, with respect to a set of lattice operations contained in Ω, be a lattice with zero element O and complemented principal ideals. Let S be an ideal of A with respect to the lattice operations. For every translation τ' such that $O\tau' = O$, let $S\tau' \subset S$ hold: then, for every $\tau \in \tilde{\Omega}_L$ such that $O\tau \in S$, $S\tau \subset S$.*

Proof. Let S be an ideal (with respect to the lattice operations) of A. Let $\tau \in \tilde{\Omega}_L$ be a translation such that $O\tau = b \in S$: let a be an element of S and let $a\tau = z$ hold. We must prove that $z \in S$.

We have that

$$O \subset a \Rightarrow O\tau \subset a\tau \Leftrightarrow b \subset z .$$

Let b' be a complement of b in the principal ideal $[O, z]$ and let we consider the derived translation $x\tau \cap b'$: then we have $O\tau \cap b' = b \cap b' = O$, from which it follows from the assumptions, that $a\tau \cap b' \in S$. Now $a\tau \cap b' =$

(9) In chain C condition (α') is always true as it can be directly seen by observing that, given an ideal S, every translation may be expressed by means of the elementary translations $a \mathrm{K} x$ $(a \in S)$, $b \cap x$ $(b \in S)$.

$= z \cap b' = b'$, then $b' \in S$; therefore $b \in S \Rightarrow b' \in S$ and we finally have $b \cap b' = z \in S$, as S is an ideal.

Thus Proposition 3 is proved.

At last we give another example in order to illustrate the fact that lattices in which Proposition 3 is not true exist. We consider the modular lattice L_6 which is obtained from the previously considered L_5 by attaching below a new element, the O (thus now we call $a_1 \cap a_2 \cap a_3$ the element which covers it. The translations $x \cup a_2$, $x \cup a_3$ do not leave the principal ideal $[O, a_1]$ invariant, so the derived translation $(x \cup a_2) \cap (x \cup a_3)$, which maps O on $a_1 \cap a_2 \cap a_3$, also maps a_1 on $I \notin [O, a_1]$ ($[O, a_1]$ is not a normal subset). On the other hand, every translation τ' such that $O\tau' = O$ can be expressed as intersections, from which follows $[O, a_1]\tau' \subset [O, a_1]$ as $[O, a_1]$ is an ideal: therefore, in the present example, if we call S the ideal $[O, a_1]$, the condition $(O\tau' = O \Rightarrow S\tau' \subset S)$ does not imply $(O\tau \in S \Rightarrow S\tau \subset S)$.

References.

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Summary

Within a framework of general algebra we give here a necessary and sufficient condition for the existence of congruences. Thus we investigate some existence properties of congruences in general algebrae with lattice operations and we apply the general proposition to a wide class of these algebrae.

Sommario

In un contesto di algebra universale si enuncia una condizione necessaria e sufficiente per l'esistenza di congruenze. Di conseguenza si esaminano alcune proprietà della esistenza di congruenze in algebre generali con operazioni di reticolo e si applica la proposizione generale ad una ampia classe di tali algebre.

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