Riv. Mat. Univ. Parma (4) 1 (1975), 39-43

S. M. Patel (*)

Conditions implying normality of spectraloid operators. (**)

The primary object of the present Note is to prove the following result: If T is a non-singular operator on a Hilbert space such that

(1) T and T^{-1} are both reduction-spectraloid,

(2) $T^{n_1}T^{*n_2}S = ST^{*m_1}T^{m_2} + K$, where 0 is not in the essential numerical range of T, n_1, n_2, m_1, m_2 are integers with $n_1 + n_2 \neq m_1 + m_2$ and K is compact, then T is a normal operator.

In what follows, H will be a separable infinite-dimensional Hilbert space. Let $\sigma(T)$, $\pi_{00}(T)$, and $\overline{W(T)}$ denote the spectrum, the set of isolated points in $\sigma(T)$ that are eigenvalues of finite multiplicity and the closure of the numerical range of T.

We write $r(T)$ and $|W(T)|$ for the spectral radius and the numerical radius of T. An operator T is called spectraloid if $r(T) = |W(T)|$. If every direct summand of T is spectraloid, then T is called reduction-spectraloid. The left essential spectrum of T, written as $\sigma_1(\hat{T})$, is the collection of all z's such that $\hat{T} - z\hat{I}$ (the image of $T - zI$ in the Calkin algebra) fails to be left regular. According to [**3**], $z \in \sigma_1(\hat{T})$ if and only if there exists a sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly in H and $\|(T - zI)x_n\| \to 0$. The numerical range of $\hat{T}$, denoted by $W_e(\hat{T})$, is called the essential numerical range of T. In ([**8**], Theorem 9) it is shown that $W_e(\hat{T}) = \bigcap_K \overline{W(T+K)}$ where the intersection is taken over all compact operators K.

(*) Indirizzo: Department of Mathematics, Sardar Patel University, Villabh Vidyanagar 388220, Via Anand, Gujarat, India.

(**) Ricevuto: 23-II-1973.

Before we state our results we quote the following as Lemmas.

Lemma 1 ([7], Theorem B). *If T is an invertible operator such that* $|W(T)|\leqslant 1$ *and* $|W(T^{-1})|\leqslant 1$, *then T is unitary.*

Lemma 2 ([5]$_1$, Theorem 1). $\partial\sigma(T)\subseteq\sigma_1(T)\,U\pi_{00}(T)$ $[\partial = \textit{boundary}]$.

It is shown in [6] that if T is a non-singular operator, and if $T^* = S^{-1}\cdot T^{-1}S$ and $0\notin\overline{W(S)}$, then T is similar to a unitary operator; in particular, if T and T^{-1} are both spectraloid, then T is a unitary operator. We generalize the second part of this result in the following

Theorem 1. *If T is a non-singular operator such that*

(1) *T and T^{-1} are both spectraloid,*

and

(2) $T^{n_1}T^{*n_2} = ST^{*m_1}T^{m_2}S^{-1}$ *and* $0\notin\overline{W(S)}$,

where n_1, n_2, m_1 and m_2 are some integers with $n_1+n_2\neq m_1+m_2$, then T is unitary.

Proof. Since T is spectraloid, there exists a complex number z in $\sigma(T)$ for which $|z| = r(T) = |W(T)|$. Then a sequence $\{x_n\}$ of unit vectors can be found such that $\|(H\text{-}zI)x_n\|\to 0$. Since $|z| = |W(T)|$, it follows that $\|(T^*-z^*I)x_n\|\to 0$ ([3], Satz 2). Then

$$|(z^{n_1}z^{*n_2}-z^{*m_1}z^{m_2})\langle Sx_n, x_n\rangle| =$$

$$= |\langle(z^{n_1}z^{*n_2}-T^{n_1}T^{*n_2}+ST^{*m_1}T^{m_2}S^{-1}-z^{*m_1}z^{m_2})Sx_n, x_n\rangle|$$

$$= |\langle Sx_n, (z^{n_2}z^{*n_1}-T^{n_2}T^{*n_1})\rangle+\langle S(T^{*m_1}T^{m_2}-z^{*m_1}z^{m_2})x_n, x_n\rangle| \leqslant \|S\|(\alpha_n+\beta_n),$$

where

$$\alpha_n = \|(T^{n_2}T^{*n_1}-z^{n_2}z^{*n_1})x_n\|, \qquad \beta_n = \|(T^{*m_1}T^{m_2}-z^{*m_1}z^{m_2})x_n\| .$$

Since $\|(T-zI)x_n\| \to 0$ and $\|(T^*-z^*I)x_n\| \to 0$, it follows that $\alpha_n \to 0$ and $\beta_n \to 0$. Consequently

$$|(z^{n_1}z^{*n_2} - z^{*m_1}z^{m_2})\langle Sx_n, x_n\rangle| \to 0 .$$

Since $0 \notin \overline{W(S)}$, we get $z^{n_1}z^{*n_2} = z^{*m_1}z^{m_2}$. Then our hypothesis that T is non-singular and $n_1+n_2 \neq m_1+m_2$ implies $|z|=1$. This shows that $|W(T)|=1$. Also, as T^{-1} is spectraloid, a similar argument yields $|W(T^{-1})|=1$. That T is a unitary operator now follows from Lemma 1.

In an attempt to extend Theorem 1 when $\hat{T}^{n_1}\hat{T}^{*n_2}$ is similar to $\hat{T}^{*m_1}\hat{T}^{m_2}$, we prove our main result.

Theorem 2. *If T is a non-singular operator such that*

(1)* *T and T^{-1} are both reduction-spectraloid,*

(2)* $T^{n_1}T^{*n_2}S = ST^{*m_1}T^{m_2} + K$ *and* $0 \notin W_e(\hat{S})$,

where $n_1+n_2 \neq m_1+m_2$ and K is a compact operator, then T is normal.

Proof. Let M be the closed linear span of all reducing eigenspaces of T. Then the restriction $T_1 = T/M$ of T to M is normal. Let $M^\perp$ be the orthogonal compliment of M. We assert that $T_2 = T/M^\perp$ is unitary; whence it will follow that $T = T_1 \oplus T_2$ is normal.

Now by our hypothesis, T_2 is spectraloid. Therefore we can choose z in $\sigma(T_2)$ such that $|z| = r(T_2) = |W(T_2)|$. Then $z \in \partial\sigma(T_2)$. By Lemma 2, it follows that either $z \in \sigma_1(\hat{T}_2)$ or $z \in \pi_{00}(T_2)$. We claim that $z \notin \pi_{00}(T_2)$. If on the contrary $z \in \pi_{00}(T_2)$, then $\operatorname{Ker}(T_2 - zI) = \operatorname{Ker}(T_2^* - z^*I)$ because $|z| = |W(T_2)|$ ([4], Satz 2). Now, by our construction $\operatorname{Ker}(T-zI) \subseteq M^\perp$ and hence $\operatorname{Ker}(T_2 - zI) = \operatorname{Ker}(T-zI)$ ([1]$_1$, Proposition 4.1). But $\operatorname{Ker}(T_2^* - z^*I) \subseteq \operatorname{Ker}(T^* - z^*I)$. Therefore $\operatorname{Ker}(T-zI) \subseteq \operatorname{Ker}(T^*-z^*I)$. This shows that $\operatorname{Ker}(T-zI)$ reduces T and hence $\operatorname{Ker}(T-zI) \subseteq M$, a contradiction. Thus $z \in \sigma_1(\hat{T}_2)$. Therefore, we can find a sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly in $M^\perp$ and $\|(T_2 - zI)x_n\| \to 0$. Again by ([4], Satz 2) $\|(T_2^* - z^*I)x_n\| \to 0$. In consequence

$$x_n \to 0 \text{ weakly in } H, \qquad \|(T-zI)x_n\| \to 0 \qquad \text{and} \qquad \|(T^*-z^*I)x_n\| \to 0 .$$

Since $0 \notin W_e(\hat{S})$, then $0 \notin \overline{W(S+K_1)}$ for some compact operator K_1. If we write $S_1 = S + K_1$, then condition (2)* reduces to

$$T^{n_1} T^{*n_2} = S_1 T^{*m_1} T^{m_2} S_1^{-1} + K_2 \qquad (K_2 = \text{compact operator}).$$

Now the compactness of K_2 implies $\|K_2^* x_n\| \to 0$. Thus a slight modification in the argument applied in Theorem 1 yields $|z| = 1$, and hence $|W(T_2)| = 1$. Similarly it can be shown that $|W(T_2^{-1})| = 1$. Again by Lemma 1, T_2 turns out to be unitary. This finishes the proof of our theorem.

Lastly, we state the following theorem which can be proved, using the techniques of previous theorems.

Theorem 3. *If T is a non-singular operator such that*

(1) $\pi_{00}(T) = \phi$,

(2) *T and T^{-1} are both spectraloid,*

(3) $T^{n_1} T^{*n_2} S = S T^{*m_1} T^{m_2} + K$ *and* $0 \notin W_e(\hat{S})$,

where $n_1 + n_2 \neq m_1 + m_2$ and K is a compact operator, then T is unitary.

Remark 1. If $n_1 + n_2 = m_1 + m_2$ then the conclusions of Theorem 1 and Theorem 2 need not be true. For example, let U be the unilateral shift. Choose a complex number z not in $\overline{W(U)}$. Then the operator $U - zI$ is a non-singular hyponormal operator such that $(U-zI)^*(U-zI) = (U^* - z^*I)(U-zI)(U-zI)^*(U^* - z^*I)^{-1}$ and $0 \notin \overline{W(U^* - z^*I)}$, thus $U - zI$ satisfies the conditions of Theorems 1, 2. However, $U - zI$ fails to be normal.

Remark 2. It is worth notice that if T is a hyponormal operator satisfying condition (2)* and moreover if $m_1 = n_1 \neq 0$ and $m_2 = n_2 = 0$, then T turns out to be normal ([4], Theorem 3.1) even if T is singular. Also for the same operator satisfying condition (2)*, if $m_2 \neq n_1$ and $m_1 \neq n_2$ (it is possible that $n_1 + n_2 = m_1 + m_2$ or $n_1 + n_2 \neq m_1 + m_2$), then $z \in \sigma_1(\hat{T})$ implies that $z^{n_2 - m_1} = z^{*m_2 - n_1}$. Infact, for $z \in \sigma_1(\hat{T})$, there exists a sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly in H and $\|(T - zI)x_n\| \to 0$, and hence by the hyponormality of $T - zI$, $\|(T^* - z^*I)x_n\| \to 0$. As argued in Theorem 2, we obtain $z^{n_2 - m_1} = z^{*m_2 - n_1}$. If $n_1 + n_2 = m_1 + m_2$, then $\sigma_1(\hat{T})$ will lie on the $n_2 - m_1$ lines through the origin. By Lemma 2, all but atmost countable number of points of $\sigma(T)$ will be on these $n_2 - m_1$ lines, and so of $\sigma(T)$. If $n_1 + n_2 \neq m_1 + m_2$, then $\sigma_1(\hat{T})$ will be finite and so again by Lemma 2, $\sigma(T)$ is atmost countable. Thus in both cases, $\sigma(T)$ has zero area. Consequently, T turns out to be normal ([7], Theorem 1). It is an open question whether these results remain true for a para-normal operator T.

Remark 3. If T^{-1} is not assumed to be spectraloid in Theorem 1, one can conclude that $|W(T)|=1$ and hence T is similar to a contraction. However T may fail to be even normal. For example, if T is a bilateral weighted shift with weights $\{\ldots 1, 2, 1, (1/2), 1, 1, 1, \ldots\}$ and S is a diagonal operator with a diagonal $\{\ldots 1, 1, 1, (1), 1/4, 1/4, 1/4, \ldots\}$, then $T^{n_1}T^{*n_2} = ST^{*m_1}T^{m_2}S^{-1}$ and $0 \notin \overline{W(S)}$ for $m_1 = 1 = n_1$ and $n_2 = m_2 = 0$. However T is not normal, although it is spectraloid.

The author wishes to express his gratitude to Dr. B. S. Yadav for his helpful suggestions.

References.

[1] S. K. Berberian: [•]$_1$ *The Weyl spectrum of an operator*, Indiana Univ. Maths. J. **20** (1970), 529-544; [•]$_2$ *Some conditions on an operator implying normality* II, Proc. Amer. Math. Soc. **26** (1970), 227-281.

[2] P. A. Fillmore, J. G. Stampfli and J. P. Williams, *On the essential numerical range, the essential spectrum and a problem of Halmos* Acta. Sci. Math. (Szged) **33** (1972), 179-192.

[3] S. Hildebrandt, *Über den numerischen Wertebereich eines operators*, Math. Ann. **163** (1966), 280-247.

[4] V. Istratescu and I. Istratescu, *On some classes of operators* - II, Math. Ann. **194** (1971), 126-134.

[5] C. R. Putnam: [•]$_1$ *The spectral properties of operators having resolvents of first order growth*, Trans. Amer. Math. Soc. **133** (1968), 505-510; [•]$_2$ *An inequality for the area of hyponormal spectra*, Math. Zeit. **116** (1970), 323-331.

[6] U. N. Singh and Kanta Mangla, *Operators with adjoints similar to their inverses*, Proc. Amer. Math. Soc. **38** (1973), 258-260.

[7] J. G. Stampfli, *Normality and the numerical range of an operator*, Bull. Amer. Math. Soc. **72** (1966), 1021-1022.

[8] J. G. Stampfli and J. P. Williams, *Growth conditions and the numerical range in Banach algebra*, Tohoku Math. J. **20** (1968), 417-424.

* * *