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Parameter-preserving groups of graphs. (**)

An *automorphism* of a graph G is a permutation α on $V(G)$ such that for all vertices u and v of G, we have that $uv \in E(G)$ if and only if $\alpha u \alpha v \in E(G)$. The set of all automorphisms of G forms a group, denoted by $\Gamma(G)$ and referred to as the *vertex-group* of G. Clearly, if $\alpha \in \Gamma(G)$ and $v \in V(G)$, then $\deg(\alpha v) = \deg v$, i.e. automorphisms preserve degrees. If G is a connected graph, the *eccentricity* $e(v)$ of a vertex v of G is the maximum distance from v among the vertices of G. It is a consequence of the definition of automorphism that if $\alpha \in \Gamma(G)$, then $e(\alpha v) = e(v)$, i.e. automorphisms preserve eccentricities. Suppose G is an r-regular graph, $v \in V(G)$, and v is incident with the edges $e_1, e_2, \ldots, e_r$ of G. Denote by μ_{ij}, $i < j$, the length of a shortest cycle of G containing e_i and e_j, where we define $\mu_{ij} = 0$ if e_i and e_j do not lie on a cycle of G. If $\mu_1, \mu_2, \ldots, \mu_{\binom{r}{2}}$ are the numbers μ_{ij} in nondecreasing order, then the *type* $\mu(v)$ of the vertex v is the $\binom{r}{2}$-tuple $(\mu_1, \mu_2, \ldots, \mu_{\binom{r}{2}})$. We observe that if $\alpha \in \Gamma(G)$, then $\mu(\alpha v) = \mu(v)$, i.e. automorphisms preserve types.

Let P be a function defined on the vertex set of a graph G. We will say that P is a *$\Gamma(G)$-preserved-function* if for each $v \in V(G)$ and each $\alpha \in \Gamma(G)$, we have $P(\alpha v) = P(v)$. As noted above, the degree function is a $\Gamma(G)$-preserved-function for every graph G and the eccentricity and type functions are $\Gamma(G)$-preserved-functions for connected graphs G and regular graphs G, respectively. If G is an arbitrary graph and P is a $\Gamma(G)$-preserved-function, we define the *P-preserving-group* $\Gamma_P(G)$ to be the group of all permutations of the vertices of G such that for each $v \in V(G)$ and for each $\alpha \in \Gamma_P(G)$, we have $P(\alpha v) = P(v)$. Then $\Gamma(G)$ is a subgroup of $\Gamma_P(G)$. In [1], BEHZAD determined neces-

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sary and sufficient conditions to have $\Gamma(G) \cong \Gamma_P(G)$ in the case $P(v) = \deg(v)$. The purpose of this paper is to extend BEHZAD's main results to a general $\Gamma(G)$-preserved-function P defined on a graph G and then determine those connected graphs G for which $\Gamma(G) \cong \Gamma_P(G)$ in the case $P(v) = e(v)$.

In order to present the following theorem, two definitions are necessary. The *neighborhood* $N_G(v)$ of a vertex v of a graph G is the set of all vertices of G which are adjacent to v. We note that if v is an isolated vertex of G, then $N_G(v) = \emptyset$. The *closed neighborhood* $\overline{N}_G(v)$ of v is $N_G(v) \cup \{v\}$.

Theorem 1. *Let P be a $\Gamma(G)$-preserved-function defined on the vertex set of a graph G. Then $\Gamma(G) \cong \Gamma_P(G)$ if and only if*

(i) *equi-P-valued vertices of G at least two of which are nonadjacent are all mutually nonadjacent and all have the same neighborhood, and*

(ii) *equi-P-valued vertices of G at least two of which are adjacent are all mutually adjacent and all have the same closed neighborhood.*

Proof. We first assume that $\Gamma(G) \cong \Gamma_P(G)$. Let w be an arbitrary vertex of G and let $S = \{v \in V(G) \mid P(v) = P(w)\}$. If $|S| = 1$, there is nothing to prove. So we assume $|S| \geq 2$. We first show that the elements of S are either mutually adjacent or mutually nonadjacent. Suppose, to the contrary, that there exist distinct vertices $u_1, u_2 \in S$ such that $u_1u_2 \notin E(G)$ and distinct vertices $v_1, v_2 \in S$ such that $v_1v_2 \in E(G)$. If $\{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset$, then we define a permutation α on $V(G)$ as follows:

$$\alpha u_1 = v_1, \qquad \alpha v_1 = u_1,$$

$$\alpha u_2 = v_2, \qquad \alpha v_2 = u_2,$$

and

$$\alpha w = w \qquad \text{for all} \qquad w \in V(G) - \{u_1, u_2, v_1, v_2\}.$$

Then $\alpha \in \Gamma_P(G)$ and $\alpha \notin \Gamma(G)$, which presents a contradiction. If $\{u_1, u_2\} \cap \cap \{v_1, v_2\} \neq \emptyset$, we may assume without loss of generality that $u_1 = v_1$ and $u_2 \neq v_2$. We define a permutation α on $V(G)$ as follows:

$$\alpha u_2 = v_2, \qquad \alpha v_2 = u_2$$

and

$$\alpha w = w \qquad \text{for all} \qquad w \in V(G) - \{u_2, v_2\}.$$

Then $\alpha \in \Gamma_P(G)$ and $\alpha \notin \Gamma(G)$, which is a contradiction. Thus, if two elements of S are adjacent, then all elements of S are mutually adjacent and if two elements of S are nonadjacent, then all elements of S are mutually nonadjacent.

We now show that if the elements of S are mutually nonadjacent, then they have the same neighborhood. Assume, to the contrary, that S contains two elements, say u_1 and u_2, such that $N_G(u_1) \neq N_G(u_2)$. We define the following permutation α on $V(G)$:

$$\alpha(u_1) = u, \qquad \alpha(u_2) = u_1,$$

and

$$\alpha(w) = w \qquad \text{for all} \qquad w \in V(G) - \{u_1, u_2\}.$$

Then $\alpha \in \Gamma_P(G)$ and $\alpha \notin \Gamma(G)$, which is a contradiction. The same argument shows that if the elements of S are mutually adjacent, then they have the same closed neighborhood.

In order to prove the converse, it suffices to show that if (i) and (ii) are satisfied, then $\Gamma_P(G)$ is a subgroup of $\Gamma(G)$. If $E(G) = \emptyset$, then $\Gamma_P(G)$ is clearly a subgroup of $\Gamma(G)$. So we may assume that $E(G) \neq \emptyset$. We first show that if $\alpha \in \Gamma_P(G)$ and $uv \in E(G)$, then $\alpha u \alpha v \in E(G)$. If $\alpha u = u$ and $\alpha v = v$, then $\alpha u \alpha v \in E(G)$. So we assume that $\alpha u \neq u$ or $\alpha v \neq v$.

Case 1. *Suppose* $P(u) = P(v)$. Since $\alpha \in \Gamma_P(G)$, we have $P(\alpha u) = P(u) = P(v) = P(\alpha v)$. Since $\alpha u \neq \alpha v$ and $uv \in E(G)$, condition (ii) implies that $\alpha u \alpha v \in E(G)$.

Case 2. *Suppose* $P(u) \neq P(v)$ *and* $\alpha u = u$ *or* $\alpha v = v$, *say the former*. Then $\alpha v \neq v$ and since $\alpha \in \Gamma_P(G)$, we have $P(\alpha v) = P(v)$. Since $\alpha u \in N_G(v)$, conditions (i) and (ii) imply that $\alpha u \in \overline{N}_G(\alpha v)$. However, $\alpha u \neq \alpha v$ so that $\alpha u \in N_G(\alpha v)$, i.e. $\alpha u \alpha v \in E(G)$.

Case 3. *Suppose* $P(u) \neq P(v)$, $\alpha u \neq u$, *and* $\alpha v \neq v$. Since $\alpha \in \Gamma_P(G)$, we have $P(\alpha u) = P(u)$ and $P(\alpha v) = P(v)$. Since $v \in N_G(u)$, conditions (i) and (ii) imply that $v \in \overline{N}_G(\alpha u)$. However, since $P(\alpha u) = P(u)$ and $P(v) \neq P(u)$, we have that $v \neq \alpha u$. Thus $v \in N_G(\alpha u)$ so that $\alpha u \in N_G(v)$. By conditions (i) and (ii), we have $\alpha u \in \overline{N}_G(\alpha v)$. Since $\alpha u \neq \alpha v$, we conclude that $\alpha u \in N_G(\alpha v)$, i.e. $\alpha u \alpha v \in E(G)$.

Corollary 1. *Let P be a $\Gamma(G)$-preserved-function defined on the vertex set of a disconnected graph G. Then $\Gamma(G) \cong \Gamma_P(G)$ if and only if*

(i) *no two components of* G, *at least one of which is nontrivial, contain equi-*P*-valued vertices, and*

(ii) *for every component* H *of* G, *we have* $\Gamma(H) \cong \Gamma_{P|V(H)}(H)$.

Corollary 2. *Let* P *be a* $\Gamma(G)$*-preserved-function defined on the vertex set of a graph* G, *where* $P(u) = P(v)$ *for all* $u, v \in V(G)$. *Then* $\Gamma(G) \cong \Gamma_P(G)$ *if and only if* G *is isomorphic to a complete graph or the complement of a complete graph.*

Corollary 3. *Let* P *be a* $\Gamma(G)$*-preserved-function defined on a graph* G *with a cutvertex* v. *If* $\Gamma(G) \cong \Gamma_P(G)$, *then there exists no vertex* $w \neq v$ *such that* $P(w) = P(v)$.

Proof. Assume, to the contrary, that G contains a vertex $w \neq v$ such that $P(w) = P(v)$. Let u and u' be two vertices of G adjacent to v which lie in different components of $G - v$. If $w \in \{u, u'\}$, say $w = u$, then $uu' \in E(G)$, contradicting the fact that u and u' lie in different components of $G - v$. Hence we must have that $w \notin \{u, u'\}$. But then w is adjacent to both u and u', again contradicting the fact that u and u' lie in different components of $G - v$.

For an arbitrary graph G, the determination of $\Gamma(G)$ is, in general, a tedious procedure. If, however, there exists a $\Gamma(G)$-preserved-function P defined on the vertex set of G such that $\Gamma(G) \cong \Gamma_P(G)$, then $\Gamma(G)$ can easily be produced. Let $V_1, V_2, \ldots, V_k$ be the partition of $V(G)$ defined by: $v, w \in V_j$ $(1 \leqslant j \leqslant k)$ if and only if $P(v) = P(w)$. Then it is easily verified that $\Gamma_P(G) \cong S_{|V_1|} \times S_{|V_2|} \times \ldots \times S_{|V_k|}$, where S_n denotes the symmetric group on n objects.

We now restrict our attention to connected graphs G and the $\Gamma(G)$-preserved-function P defined by $P(v) = e(v)$. Some preliminary definitions are needed in order to present a characterization of those connected graphs G for which $\Gamma(G) \cong \Gamma_e(G)$. We define K_p to be the graph of order p in which every pair of vertices is adjacent and $\bar{K}_p$ to be the graph of order p with no edges. Let G_1 and G_2 be graphs with $V(G_1) \cap V(G_2) = \emptyset$. Then $G_1 + G_2$ is defined to be that graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2) \cup \{v_1 v_2 \mid v_i \in V(G_i),\ i = 1, 2\}$.

Theorem 2. *Let* G *be a connected graph of order* p. *Then* $\Gamma(G) \cong \Gamma_e(G)$ *if and only if* G *is isomorphic to* K_p *or* G *is isomorphic to* $K_m + \bar{K}_n$, *for some* m *and* n *satisfying* $m + n = p$ *and* $n \geqslant 2$.

Proof. By Theorem 1, if G is isomorphic to K_p or G is isomorphic to $K_m + \bar{K}_n$, then $\Gamma(G) \cong \Gamma_e(G)$.

In order to verify the converse, we let G be a connected graph of order p such that $\Gamma(G) \cong \Gamma_e(G)$. We first observe that diam $G \leqslant 2$; for otherwise, there exists a longest distance path $P: u_0, u_1, \ldots, u_k$, where $k = \operatorname{diam} G \geqslant 3$. Then $e(u_0) = k = e(u_k)$. Since $k \geqslant 3$, neither $u_0 u_{k-1}$ nor $u_0 u_k$ are edges of G. Thus by Theorem 1, $\Gamma(G) \neq \Gamma_e(G)$, which presents a contradiction so that diam $G \leqslant 2$.

If diam $G = 0$ or diam $G = 1$, then G is isomorphic to K_p. So we may assume that diam $G = 2$. Let $S = \{v \in V(G) \mid e(v) = 1\}$ and let $T = \{v \in V(G) \mid e(v) = 2\}$. The set T is not empty since diam $G = 2$. Moreover, $S \neq \emptyset$; for otherwise, every vertex of G has eccentricity 2 in G. By Theorem 1, the vertices of G are mutually adjacent, implying that diam $G = 1$ which is a contradiction. Let $m = |S|$ and $n = |T|$ For $v \in S$ and $w \in V(G) - \{v\}$, we have that $vw \in E(G)$ since $e(v) = 1$. Let $z \in T$. Since $e(z) = 2$, there exists a vertex z' (necessarily in T) such that z and z' are nonadjacent. Thus by Theorem 1, the vertices of T are mutually nonadjacent. Hence G is isomorphic to $K_m + \bar{K}_n$, where $m + n = p$ and $n \geqslant 2$.

Reference.

[1] M. Behzad, *The degree preserving group of a graph*, Riv. Mat. Univ. Parma (2) **11** (1970), 307-311.

Abstract.

For a graph G, $\Gamma(G)$ is the group of all automorphisms of G. A function P (on $V(G)$) is a $\Gamma(G)$-preserved-function if $P(\alpha v) = P(v)$ for each $v \in V(G)$ and $\alpha \in \Gamma(G)$. For such a function, $\Gamma_P(G)$ is the group of all permutations of $V(G)$ such that $P(\alpha v) = P(v)$ for each $v \in V(G)$ and $\alpha \in \Gamma_P(G)$. Necessary and sufficient conditions are established in order to have $\Gamma(G) \cong \Gamma_P(G)$, and a specialized result is given for one particular function P.

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