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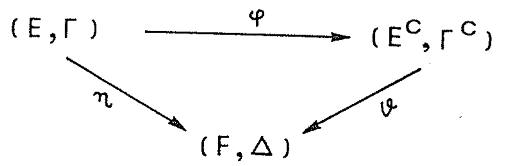
On the completion of a b-space. (**)

This paper concerns the structure of the completion of a b-space (which may have very pathological properties). We shall represent a b-space by a couple (E, Γ) where E stands for a real or complex vector space and Γ is a class of subsets of E that satisfies L. Waelbroeck's axioms for a separated convex bornology. A b-space is said to be complete iff its bornology has a basis which consists of completant convex balanced sets.

A completion of a b-space (E, Γ) will be a couple $((E^c, \Gamma^c), \varphi)$ of a complete b-space (E^c, Γ^c) and a bounded linear map $\varphi: (E, \Gamma) \to (E^c, \Gamma^c)$ which has the following universal property: for every complete b-space (F, Δ) and every bounded linear map $\eta: (E, \Gamma) \to (F, \Delta)$ there exists a unique bounded linear map θ that makes the diagram

(I)



commutative.

The map φ needs not be injective (see [3]) and, a fortiori, it may happen that φ is not a b-isomorphism onto the image. Of course, the behaviour of the completion depends upon the relationships which can exist between any two Minkowsky functionals defined by convex balanced bounded sets. We can precise this point by using the notion of *strong* and *weak concordance of norms* given in [2], chapitre IV, 3.2, as follows: *φ is a is b-isomorphism onto the image*

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iff the b-structure Γ on E can be generated by a subclass consisting of convex balanced sets such that the respective Minkowsky functionals are in strong concordance. The weak concordance of the Minkowsky functionals of the sets of a generating subclass of Γ does imply that φ is injective, but the converse is not true.

Before stating Theorem 1, let us illustrate the last part of the preceding assertion by a counterexample: denote $\boldsymbol{N} = \{1, 2 ..\}$ and let E be the $\boldsymbol{C}$-vector space freely generated by $\boldsymbol{N} \cup \boldsymbol{N}^2$ (every element of E is a finite linear conbination of vectors of type $\langle s \rangle$ and $\langle i, s \rangle$, with $i, s \in \boldsymbol{N}$) and let Γ be the (separated) convex bornology on E generated by the subsets: $A = \{n \cdot \sum_{n \leqslant i \leqslant n+p} \langle i, s \rangle \mid n, s \in \boldsymbol{N},\ \ p \in \boldsymbol{N} \cup \{0\}\}$; $B_s = \{n \sum_{1 \leqslant i \leqslant n} \langle i, s \rangle - \langle s \rangle) \mid n \in \boldsymbol{N}\}$, $(s \in \boldsymbol{N})$; $C = \{\langle s \rangle \mid s \in \boldsymbol{N}\}$. If $((E^c, \Gamma^c), \varphi)$ is a completion of (E, Γ), then φ is injective but Γ cannot be generated by a subclass with weakly concordant Minkowsky functionals.

Theorem 1. *Let (E, Γ) be a b-space and $((E^c, \Gamma^c), \varphi)$ its completion. The map φ is a b-isomorphism onto the image iff (E, Γ) is a b-subspace of a complete b-space.*

Proof. The condition is obviously a necessary one. On the other hand let us suppose (E, Γ) is a b-subspace of a complete b-space (F, Δ); by definition of completion there exists a (unique) bounded linear map θ which makes the diagram (I) commutative where η must be replaced by the inclusion map i. It is easily seen that φ is injective; furthermore, if $B \in \Gamma^c$ then $\theta(B) \in \Delta$, so that $i^{-1}(\theta(B)) \in \Gamma$; but $i^{-1}(\theta(B)) \supseteq \varphi^{-1}(B)$. The proof is finished.

It can be remarked that, in contrast with the completion of uniform structures, the completion of (E, Γ) is not necessarily b-isomorphic to $((\bar{E}^{\Delta}, \Delta \,|\, \bar{E}^{\Delta}), i)$, where $\bar{E}^{\Delta}$ stands for the b-closure [1] of E in (F, Δ) (see [3], 4.6 Rmk. 3); more precisely, we have

Theorem 2. *Let (E, Γ) be a subspace of a complete b-space (F, Δ), and i the inclusion map. Then there exists a finest b-space structure on F among those that are complete and induce Γ on E. Denote it by Δ_0; the completion of (E, Γ) is b-isomorphic to $((\bar{E}^{\Delta_0}, \Delta_0 \,|\, \bar{E}^{\Delta_0}), i)$ where $\bar{E}^{\Delta_0}$ stands for the b-closure of E in (F, Δ_0).*

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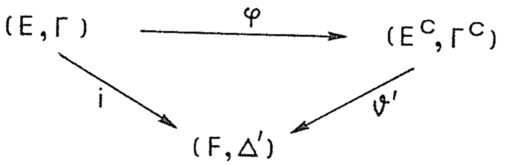
(1) A subset S of a b-space is said to be b-closed iff S contains the limit of every sequence $(x_n)_{n \in \boldsymbol{N}}$, $x_n \in S$ for every $n \in \boldsymbol{N}$, which converges in the sense of Mackey. The b-closure of a subset T is, by definition, the intersection of all the b-closed subsets containing T.

Proof. Denote by $(\mathscr{H}, \leqslant)$ the set $\mathscr{H}$ of the b-space structures on F that are complete, are finer than Δ and induce Γ on E, endowed with the fineness ordering $\leqslant$, i.e. $\Delta' \leqslant \Delta''$ iff the identical map of (F, Δ'') into (F, Δ') is bounded. $(\mathscr{H}, \leqslant)$ is a non-empty directed set.

Let $((E^c, \Gamma^c), \varphi)$ be the completion of (E, Γ). By definition of completion, there exists a unique bounded linear map θ which makes the diagram (I), with η replaced by i, commutative. We claim that θ does not change by taking $\Delta' \in \mathscr{H}$: the proof will consist in verifying that if $\Delta' \in \mathscr{H}$ and θ' is the unique bounded linear map which makes the diagram

(II)

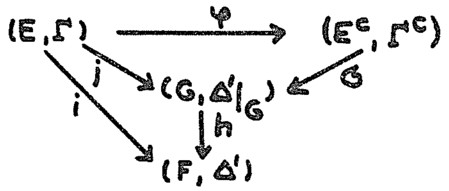


commutative, then $\theta = \theta'$. Indeed, if Id_F is the identical map of (F, Δ') onto (F, Δ), then $(Id_F \circ \theta') \circ \varphi = Id_F \circ (\theta' \circ \varphi) = Id_F \circ i = i$, so that $Id_F \circ \theta' = \theta$ by definition of completion (θ is the unique bounded linear map s.t. $\theta \circ \varphi = i$).

Let us show that θ is injective. Suppose $x \in E^c$ and $\theta(x) = 0$. We can choose a sequence $(x_n)_{n \in N}$, $x_n \in E$ for every $n \in N$, s.t. $(\varphi(x_n)) \to x$ for Γ^c (this possibility results from the description of the completion see [2], p. 36 or [3], p. 434) and consequently $((\theta \circ \varphi)(x_n)) \to 0$ for Δ; as $\theta \circ \varphi = i$ and i is by hypothesis a b-isomorphism onto the image, it follows that $(x_n) \to 0$ for Γ, so that $(\varphi(x_n)) \to 0$ for Γ^c and we get $x = 0$ as we wanted to see.

Now, given any $\Delta' \in \mathscr{H}$ put $\bar{E}^{\Delta'}$ for the b-closure of E in (F, Δ'); we have $\theta(E^c) \subseteq \bar{E}^{\Delta'}$. This assertion follows from a general argument: if $(G, \Delta'|_G)$ is a b-closed, b-subspace of (F, Δ') containing E, then $(G, \Delta'|_G)$ is complete, so that there exists a unique bounded linear map σ which makes the diagram

(III)



(where j and h are inclusion maps) commutative. As $(h \circ \sigma) \circ \varphi = h \circ (\sigma \circ \varphi) = h \circ j = i$, by definition of completion we obtain $\theta = h \circ \sigma$ and as a particular consequence $\theta(E^c) \subseteq G$.

If we put $\theta(\Gamma^c) \equiv \{\theta(B) \mid B \in \Gamma^c\}$ and denote by $\widetilde{\theta(\Gamma^c)}$ the b-space structure on F generated by $\theta(\Gamma^c)$ (i.e. the finest b-space structure on F for which the map θ is bounded), then $\widetilde{\theta(\Gamma^c)} \geqslant \Delta'$ for every $\Delta' \in \mathscr{H}$. As (E^c, Γ^c) is complete, $(F, \widetilde{\theta(\Gamma^c)})$ is also complete. To finish, we must now verify that $\widetilde{\theta(\Gamma^c)}$ induces Γ on E; this follows from the following chain of inequalities:

$$\Gamma \geqslant \varphi^{-1}(\Gamma^c) = \varphi^{-1} \circ \theta^{-1} \circ \theta(\Gamma^c) = i^{-1}(\theta(\Gamma^c)) = i^{-1}(\widetilde{\theta(\Gamma^c)}) \geqslant i^{-1}(\Delta) = \Gamma .$$

From the above arguments $\widetilde{\theta(\Gamma^c)}$ results to be maximum in $(\mathscr{H}, \leqslant)$; we write $\Delta_0 = \widetilde{\theta(\Gamma^c)}$.

The map θ may be considered as a b-isomorphism of (E^c, Γ^c) onto $(\theta(E^c), \Delta_0|_{\theta(E^c)})$; this shows that the latter b-space is complete, so that $\theta(E^c)$ is b-closed in (F, Δ_0); as $\theta(E^c)$ contains E we deduce $\theta(E^c) \supseteq \bar{E}^{\Delta_0}$. This fact and the relation $\theta(E^c) \subseteq \bar{E}^{\Delta}$, $\Delta' \in \mathscr{H}$, already obtained, imply $\theta(E^c) = \bar{E}^{\Delta_0}$. Thus, we can conclude that $(\bar{E}^{\Delta_0}, \Delta_0|_{\bar{E}^{\Delta_0}})$ is b-isomorphic to (E^c, Γ^c) by θ and the proof is finished.

Corollary. *Let (E, Γ) be a dense b-subspace of a complete b-space (F, Δ), and i the inclusion map. Then the completion of (E, Γ) is b-isomorphic to $((F, \Delta), i)$ iff Δ is the finest b-space structure on F which is complete and induces Γ on E. As a necessary condition, every point of F is the limit in the sense of Mackey of a sequence contained in E.*

References.

[1] H. Hogbe-Nlend, *Complétion, Tenseurs et Nucléarité en Bornologie*, J. Math. Pures Appl. **49** (1970), 193-288.

[2] H. Hogbe-Nlend, *Théorie des Bornologies et Applications*, Springer Verlag, Berlin 1971.

[3] A. V. Ferreira, *Some remarks on b-spaces*, Portugaliae Mathematica **26** (1967), 421-447.

Riassunto.

Il completamento di uno spazio vettoriale munito di una bornologia nel senso di L. Waelbroeck ha in genere delle proprietà patologiche in netto contrasto con la natura piuttosto regolare dell'immersione di uno spazio uniforme nel suo completato; questo lavoro concerne proprietà di regolarità del completamento bornologico.

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