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Submanifolds of almost hermitian manifolds and normal connection. (**)

Introduction.

In the theory of almost hermitian manifolds there exist for some special classes interesting identities for the RIEMANN-CHRISTOFFEL curvature tensor [5], $[4]_3$. In particular, the manifolds such that $R(X, Y, Z, W) = R(X, Y, JZ, JW)$ are called *para-Kähler manifolds* [8] or *F-spaces* [9]. A lot of properties for Kähler manifolds are still valid for this more general class of almost hermitian manifolds.

On the other hand, complex submanifolds of Kähler and nearly Kähler manifolds are such that the second fundamental form is *complex bilinear* [7], $[11]_3$ and this property implies that the submanifolds are minimal.

In this paper we treat complex submanifolds of para-Kähler manifolds such that they have complex bilinear second fundamental form and we call them *σ-submanifolds*. In particular we consider the associated *normal connection* and show that some theorems proved in [2] for Kähler manifolds still hold for this more general class.

1. – Let M be an n-dimensional C^∞ riemannian manifold with LEVI-CIVITA connection ∇. Then the curvature tensor R of M is given by

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

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for any $X, Y \in \mathscr{X}(M)$ where $\mathscr{X}(M)$ is the LIE algebra of C^∞ vector fields on M. Further, let $\{E_i\}$ be a local orthonormal frame field on M. Then the *Ricci tensor* $S(X, Y)$ is defined by

$$S(X, Y) = \sum_{i=1}^{n} R(X, E_i, Y, E_i) ,$$

where $R(X, E_i, Y, E_i) = g(R(X, E_i)Y, E_i)$ and g is the metric tensor of M.

Let $x: M \to \tilde{M}$ be an isometric immersion of M into an m-dimensional riemannian manifold $\tilde{M}$ with connection $\tilde{\nabla}$ and metric tensor $\tilde{g}$. Then we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) ,$$

where $\nabla_X Y$ denotes the component of $\tilde{\nabla}_X Y$ tangent to M and $X, Y \in \mathscr{X}(M)$. σ is a symmetric covariant tensor field of degree 2 with values in $\mathscr{X}(M)^\perp$. We have further

$$\tilde{\nabla}_X N = -A_N X + D_X N ,$$

where: N is a normal vector field. $-A_N X$ (resp. $D_X N$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X N$. D is the linear connection in the normal bundle $T(M)^\perp$ and A is a cross-section of a vector bundle $\operatorname{Hom}(T(M)^\perp, S(M))$ where $S(M)$ is the bundle whose fibre at each point is the space of symmetric linear transformations of $T_m(M) \to T_m(M)$, $m \in M$, i.e. for any normal vector $N \in T_m(M)^\perp$, $A_N: T_m(M) \to T_m(M)$. We have

$$\tilde{g}(N, \sigma(X, Y)) = g(A_N X, Y) = g(X, A_N Y) .$$

σ and A are called both the *second fundamental form* of M and D is called the *normal connection*.

A local normal vector field $N \neq 0$ is called a *parallel section* if $DN = 0$. Let $R^\perp$ be the curvature tensor associated with D, i.e. $R^\perp(X, Y) = D_{[X,Y]} - [D_X, D_Y]$. Then the normal connection is *flat* if $R^\perp$ vanishes identically. The normal connection is flat if the (real) codimension is one and if the (real) codimension is higher, then the normal connection is not flat in general.

The *equations of Gauss and Ricci* are given respectively by

$$(1) \qquad \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \tilde{g}\{\sigma(X, W), \sigma(Y, Z)\} - \tilde{g}\{\sigma(X, Z), \sigma(Y, W)\}$$

and

$$(2) \qquad \{\tilde{R}(X, Y)N\}^\perp = R^\perp(X, Y)N - \sigma(A_N X, Y) + \sigma(X, A_N Y)$$

or

$$\tilde{R}(X, Y, N, N') = R^{\perp}(X, Y, N, N') + g([A_N, A_{N'}]X, Y) \tag{2'}$$

where $X, Y, Z, W \in \mathscr{X}(M)$ and $N, N' \in \mathscr{X}(M)^{\perp}$.

2. – Let $(\tilde{M}, g, J)$ be a C^∞ manifold which is *almost hermitian*, that is, the tangent bundle has an almost complex structure J and a riemannian metric g such that $g(JX, JY) = g(X, Y)$ for all $X, Y \in \mathscr{X}(\tilde{M})$. Then $\dim \tilde{M} = m = 2p$ and $\tilde{M}$ is orientable.

Moreover, let M be an almost hermitian manifold of dimension $n = 2q$ and f a complex immersion of M into $\tilde{M}$. Then, for each $m \in M$ we identify the tangent space $T_m(M)$ with $f_*(T_m(M)) \subset T_{f(m)}(\tilde{M})$ by means of f_*. Since $f_* \circ g = g'$ and $J \circ f_* = f_* \circ J'$ where g' and J' are the hermitian metric and the almost complex structure of M respectively, g' and J' are respectively identified with the restrictions of the structures g and J to the subspace $f_*(T_m(M))$.

Let $K(X, Y)$ denote the sectional curvature for the 2-plane spanned by $X, Y \in T_m(M)$ and $H(X)$ the holomorphic sectional curvature of the 2-plane spanned by X and JX.

An almost hermitian manifold $\tilde{M}$ such that

$$\tilde{\nabla}_X(J)\,Y = 0\,, \qquad X, Y \in \mathscr{X}(\tilde{M})$$

is a *Kähler manifold* and a *nearly Kähler manifold* [4]₁, [4]₂ (an *almost Tachibana space* or *K-space* [10]) is defined by

$$\tilde{\nabla}_X(J)\,X = 0 \qquad \text{or equivalently} \qquad \tilde{\nabla}_X(J)\,Y + \tilde{\nabla}_Y(J)\,X = 0$$

for all $X, Y \in \mathscr{X}(\tilde{M})$. We proved in [11]₃ that if M is a complex submanifold of a Kähler or nearly Kähler manifold, then the second fundamental form of M always satisfies

$$\sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y) \qquad \text{for all } X, Y \in \mathscr{X}(M)\,. \tag{3}$$

This means that σ is *complex bilinear* [7].

In what follows we only consider complex submanifolds of an almost hermitian manifold $\tilde{M}$ such that (3) is satisfied and therefore we give the following definition.

Definition. *A σ-submanifold M of an almost hermitian manifold is a complex submanifold satisfying* (3).

It is easy to prove the following theorems.

Theorem 1. *A σ-submanifold of an almost hermitian manifold is minimal.*

Theorem 2. *The condition* (3) *is equivalent with*

$$A_{JN}X = JA_N X\,, \qquad A_N JX + JA_N X = 0 \tag{4}$$

for all $X \in \mathscr{X}(M)$ and $N \in \mathscr{X}(M)^{\perp}$.

3. – Let $\tilde{M}$ be an almost hermitian manifold and M a σ-submanifold. Suppose that N is a unit parallel section in the normal bundle. We have $DN = 0$. Then $R^{\perp}(X, Y)N = 0$ for all $X, Y \in \mathscr{X}(M)$. From the equation of Ricci we find

$$\tilde{R}(X, Y, N, JN) = g([A_N, A_{JN}]X, Y)$$

and with (4) we get

$$\tilde{R}(X, Y, N, JN) = -2g(JA_N^2 X, Y)\,. \tag{5}$$

Let now $H(X, N)$ denote the *holomorphic bisectional curvature* [**3**] for the pair (X, N):

$$H(X, N) = \tilde{R}(X, JX, N, JN)\, g^{-1}(X, X)\, g^{-1}(N, N)\,.$$

If X is a unit tangent vector of M, then it follows from (5)

$$H(X, N) = -g(A_N X, A_N X) \tag{6}$$

and hence

Theorem 3. *Let M be a σ-submanifold of an almost hermitian manifold $\tilde{M}$. If there is a unit tangent vector X such that for all unit normal vectors N the holomorphic bisectional curvatures $H(X, N)$ are positive, then the normal bundle admits no parallel section.*

A unit section in the normal bundle such that $A_N = 0$ is called a *geodesic section*. It follows then from (6)

Theorem 4. *Let M and $\tilde{M}$ be as in Theorem* 1 *and let N be a unit parallel section N in the normal bundle. Then for all tangent vectors X the holomorphic bisectional curvature $H(X, N)$ vanishes if and only if N is a geodesic section.*

4. – Now we consider an almost hermitian manifold $\tilde{M}$ and the tensor λ on $\tilde{M}$ defined by

$$\lambda(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) - \tilde{R}(X, Y, JZ, JW)$$

for $X, Y, Z, W \in \mathscr{X}(M)$. A manifold $\tilde{M}$ such that

$$\lambda(X, Y, Z, W) = 0 \tag{7}$$

is called a *para-Kähler manifold* [**8**] and since (7) is equivalent with

$$\tilde{R}(X, Y)\cdot J = 0 \tag{7'}$$

such a manifold $\tilde{M}$ is an *F-space* [**9**]. In (7′) $\tilde{R}(X, Y)$ operates as a derivation on the almost complex structure J.

It is interesting to note that for an F-space we have

$$\tilde{R}(X, Y, Z, W) = \tilde{R}(JX, JY, JZ, JW)$$

and since this relation defines an *RK-manifold* [**11**]$_1$, [**11**]$_2$, [**11**]$_3$ we obtain that *an F-space is an RK-manifold.* Hence we have the following identities:

$$K(X, Y) = K(JX, JY)\,, \qquad K(X, JY) = K(JX, Y)\,,$$

$$S(X, Y) = S(JX, JY)\,, \qquad S(X, JY) + S(JX, Y) = 0\,.$$

Remark. H. YANAMOTO has given in [**12**] an example of a nonkählerian quasi-Kähler manifold (or *0-space) satisfying (7) and in [**9**] it is proved that a para-Kähler manifold with nonzero constant holomorphic sectional curvature is a Kähler manifold.

A *quasi-Kähler* manifold or **0-space* is an almost hermitian manifold such that

$$\tilde{\nabla}_X(J)\,Y + \tilde{\nabla}_{JX}(J)\,JY = 0$$

for all $X, Y \in \mathscr{X}(\tilde{M})$ [**4**]$_1$, [**6**].

It follows easily from (1) and (3):

Theorem 5. *A σ-submanifold of a para-Kähler manifold $\tilde{M}$ is also para-Kählerian.*

5. – In what follows we consider now σ-submanifolds of para-Kähler manifolds in relation with the normal connection. First we prove

Theorem 6. *Let M^n be a σ-submanifold of a para-Kähler manifold $\tilde{M}^m$ such that the normal connection is flat. Then, the Ricci tensors S and $\tilde{S}$ of M^n and $\tilde{M}^m$ satisfy the relation $S(X, Y) = \tilde{S}(X, Y)$ for all $X, Y \in \mathscr{X}(M^n)$.*

Proof. It follows from the Gauss equation and from (3) that

$$\tilde{S}(X, Y) = \sum_{\alpha=1}^{p-q} \{\tilde{R}(X, E_\alpha, Y, E_\alpha) + \tilde{R}(X, JE_\alpha, Y, JE_\alpha)\} + \\ + S(X, Y) + 2\sum_{i=1}^{q} g\{\sigma(X, E_i), \sigma(Y, E_i)\} \tag{8}$$

if $\dim \tilde{M} = m = 2p$, $\dim M = n = 2q$ and $\{E_\lambda, JE_\lambda\}$ denote an orthonormal local frame such that i (resp. α) denotes the tangential (resp. normal) vectors of the frame.

Suppose now that the normal connection is flat. Then, by Proposition 1.1 in ([**2**], p. 99), there exist locally $m-n$ mutually orthogonal unit normal vector fields N_r, $r = 1, 2, \ldots, m-n$, such that $DN_r = 0$. In what follows we suppose

$$E_\alpha = N_\alpha, \qquad \alpha = 1, 2, \ldots, p-q.$$

Hence we get from (5)

$$\tilde{R}(X, Y, E_\alpha, JE_\alpha) = -2g(JA_\alpha^2 X, Y) \tag{9}$$

where $A_\alpha = A_{E_\alpha}$.

Using the first Bianchi identity and the definition of a para-Kähler manifold we have

$$\tilde{R}(X, E_\alpha, Y, E_\alpha) = \tilde{R}(X, E_\alpha, JY, JE_\alpha) = \\ = -\tilde{R}(E_\alpha, JY, X, JE_\alpha) + \tilde{R}(X, JY, E_\alpha, JE_\alpha),$$

$$\tilde{R}(X, JE_\alpha, Y, JE_\alpha) = -\tilde{R}(X, JE_\alpha, JY, E_\alpha)$$

and so

$$\tilde{R}(X, E_\alpha, Y, E_\alpha) + \tilde{R}(X, JE_\alpha, Y, JE_\alpha) = \tilde{R}(X, JY, E_\alpha, JE_\alpha). \tag{10}$$

From (8), (9) and (10) it follows then

$$(11)\qquad \tilde{S}(X, Y) = S(X, Y) - 2\sum_{\alpha=1}^{p-q} g(A_\alpha^2 X, Y) + 2\sum_{i=1}^{q} g\{\sigma(X, E_i), \sigma(Y, E_i)\}$$

and with (3) we may find

$$(12)\qquad 2\sum_{i=1}^{q} g\{\sigma(X, E_i), \sigma(Y, E_i)\} = 2\sum_{\alpha=1}^{p-q} g(A_\alpha^2 X, Y)\,.$$

Combining (11) and (12) we find the required result.

We prove now that the converse of Theorem 6 holds if the complex codimension $p-q$ is one.

Theorem 7. *Let M^n be a hypersurface of a para-Kähler manifold $\tilde{M}^{n+2}$. Then the normal connection is flat if and only if $\tilde{S}(X, Y) = S(X, Y)$ for all $X, Y \in \mathscr{X}(M^n)$.*

Proof. That the condition is necessary is proved in Theorem 6.

Suppose then $\tilde{S}(X, Y) = S(X, Y)$ for all $X, Y \in \mathscr{X}(M^n)$. It follows from (8), (10) and (12)

$$\tilde{R}(X, JY, E_{n+1}, JE_{n+1}) = -2g(A_{n+1}^2 X, Y)$$

for any normal vector field E_{n+1}. Hence, this implies with (2′) and (4)

$$R^\perp(X, JY, E_{n+1}, JE_{n+1}) = 0$$

and from this we see that $R^\perp = 0$, i.e. the normal connection is flat.

Finally, we wish to prove a corollary from Theorem 6 and one of Theorem 7. Therefore, we consider a para-Kähler manifold $\tilde{M}^m$ such that this manifold is an *Einstein space*, i.e., there exists a function $\tilde{\varrho}$ on $\tilde{M}^m$ such that $\tilde{S}(X, Y) = \tilde{\varrho} g(X, Y)$ for $X, Y \in \mathscr{X}(\tilde{M})$. The function $\tilde{\varrho}$ is called the *scalar curvature* of $\tilde{M}^m$ and it is well known that ϱ is constant if $m \geqslant 3$.

It follows at once Theorem 6 and Theorem 7:

Corollary 8. *Let M^n be a σ-submanifold of a para-Kähler Einstein space $\tilde{M}^m$. If the normal connection is flat, then M^n is also an Einstein space. Moreover, M^n and $\tilde{M}^m$ have the same scalar curvature.*

Corollary 9. *Let M^n be a σ-hypersurface of a para-Kähler manifold $\tilde{M}^{n+2}$ such that M^n and $\tilde{M}^{n+2}$ are both Einstein spaces. If the two spaces have the same scalar curvature, then the normal connection is flat.*

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