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On integrals involving H-function of Fox. (**)

1. - Introduction.

The H-function introduced by Fox ([2], p. 408) will be represented and defined as follows:

(1.1)
$$\overset{m,n}{\underset{p,q}{H}}\left[x\,\middle|\,\begin{matrix} a_p,\ \alpha_p \\ b_q,\ \beta_q\end{matrix}\right]=$$

$$=(2\pi i)^{-1}\int_L \frac{\Gamma[(b_m)-(\beta_m)\xi]\,\Gamma[1-(a_n)+(\alpha_n)\xi]}{\Gamma[1-(b_{m+1,q})+(\beta_{m+1,q})\xi]\,\Gamma[(\alpha_{n+1,p})-(\alpha_{n+1,p})\xi]}\,x^{\xi}\,\mathrm{d}\xi\,,$$

where x is not equal to zero, and an empty product is interpreted as unity; p, q, m, n are integers satisfying

$$1\leqslant m\leqslant q\,,\qquad 0\leqslant n\leqslant p\,,$$

α_j $(j=1,\dots,p)$, β_j $(j=1,\dots,q)$ are positive numbers and a_j $(j=1,\dots,p)$, b_j $(j=1,\dots,q)$, are complex numbers and that no pole of $\Gamma(b_h-\beta_h\xi)$, $(h=1, \dots, m)$ coincides with any pole of $\Gamma(1-a_j+\alpha_j\xi)$, $(j=1,\dots,n)$, i.e.

(1.2) $$\alpha_i(b_h+\nu)\neq(a_i-\eta-1)\beta_h\,,\quad(\nu,\eta=0,1,\dots;\ h=1,\dots,m;\ i=1,\dots,Tn).$$

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Further the contour L runs from $\sigma - i\infty$ to $\sigma + i\infty$ and that the points

$$(1.3) \qquad \xi = (b_h + \nu)/\beta_h \qquad (h = 1, \dots, m;\ \nu = 0, 1, \dots)\,,$$

which are poles of $\Gamma(b_h - \beta_h \xi)$, lie to the right and the points

$$(1.4) \qquad \xi = (a_i - \eta - 1)\,|\,\alpha_i \qquad (i = 1, \dots, n;\ \eta = 0, 1, \dots)\,,$$

which are poles of $\Gamma(1 - a_i + \alpha_i \xi)$, lie to the left L. Such a contour is possible on account of (1.2).

The following notations will be used throughout the present paper:

The symbol $(\in_{m,p})$ denotes the sequence of $p - m + 1$ parameters $\in_m$, $\in_{m+1}, \dots, \in_p$, but when $m = 1$, we shall denote it by $(\in_p)$ instead of $(\in_{1,p})$.

As usual, $\Gamma[(a_A)]$ denotes $\prod_{r=1}^{A} \Gamma(a_r)$. Also, the symbol $\Delta_n[m;\alpha]$ denotes $\alpha|m$, $(\alpha+1)|m$, ..., $(\alpha + n - 1)|m$ parameters but when $m = n$, we shall denote it by $\Delta[m;\alpha]$ instead of $\Delta_m[m;\alpha]$.

2. - Evaluation of the integral.

In this section, a generalisation of infinite integrals involving the H-function due to Fox, is evaluated on the basis of $[3]_1$. Thus, it should include various integrals formulae as special cases as it involves the H-function of Fox [2].

Consider a function $(m_1 + n_2 \leqslant n_1 + m_2)$

$$\varphi(t) = (2\pi i)^{-1} \int_L \frac{\Gamma[(a_{n_1}) + (A_{n_1})s]}{\Gamma[(\delta_{m_1}) + (A_{m_1})s]} \times \frac{\Gamma[(b_{n_2}) - (B_{n_2})s]}{\Gamma[(\eta_{m_2}) - (B_{m_2})s]} \times t^{-s}\,ds\,,$$

which is, in fact, the sum of n_1 generalised hypergeometric series of the type ${}_{m_1+n_2}F_{m_2+n_1-1}$.

By Mellin inversion formula

$$(1.5) \qquad \frac{\Gamma[(a_{n_1}) + (A_{n_1})s]}{\Gamma[(\delta_{m_1}) + (A_{m_1})s]} \times \frac{\Gamma[(b_{n_2}) - (B_{n_2})s]}{\Gamma[(\eta_{m_2}) - (B_{m_2})s]} = \int_0^\infty \varphi(t)\, t^{s-1}\,dt\,.$$

The generalised integral is

$$\int_0^\infty t^{2\nu-1}\varphi(t)\,\overset{m',n'}{\underset{p,q}{H}}\left[zt^{-2n}\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right]\mathrm{d}t = \tag{1.6}$$

$$= \left((2\pi)^{(\frac{1}{2}-n)(n_1+n_2-m_1-m_2)}\right)\times\left((2n)^{\sum_1^{n_1}(a_q)+\sum_1^{n_2}(b_q)}\right)\times$$

$$\times\left((2n)^{-\sum_1^{m_2}(\delta_q)-\sum_1^{m_2}(\eta_q)+2\nu(n_1-n_2-m_1+m_2)-\frac{1}{2}(n_1+n_2-m_1-m_2)}\right)\times$$

$$\times\left(\overset{M,N}{\underset{P,Q}{H}}\left[(2n)^{2n(n_2-n_1+m_1-m_2)}z\,\middle|\,I_1\right]\right),$$

where $M = m'+2nn_1$, $N = n'+2nn_2$, $P = 2nn_2+p+2nm_1$, $Q = 2nn_1+q+$ $+2nm_2$, n is a positive integer and

$$I_1 = \begin{bmatrix}\Delta_{2n}[-2n;\,R],\,(\alpha_p,\alpha_p),\,\Delta[2n;\,R']\\ \Delta[2n;\,T],\,(b_q,\beta_q),\,\Delta_{2n}\,[-2n;\,T']\end{bmatrix}.$$

In I_1, $R = \big((b_{n_2})-2\gamma-2n,\,(B_{n_2})\big)$, $R' = \big((\delta_{m_1})+2\gamma,\,(A_{m_1})\big)$, $T = \big((\alpha_{n_1})+2\gamma,\,(A_{n_1})\big)$ and $T' = \big((n_{m_2})-2\gamma-2n,\,(B_{m_2})\big)$.

The result (1.6) can be easily established by proceeding on similar lines as in the case of the generalised integral due to Verma $\big([\mathbf{3}]_1\,(4.2)\big)$ and then using the definition of the H-function of Fox [**2**].

3. - Particular cases.

Many more integral formulae can be established from it as particular cases, some of the results are known and few may be non-existent:

(i) By taking $(A_{n_1}) = k$, $(B_{n_2}) = k$, $(\alpha_p) = k$, $(\beta_q) = k$, $(A_{m_1}) = k$, $(B_{m_2}) = k$ and then using the result

$$\overset{m,n}{\underset{p,q}{H}}\left[x\left|\begin{matrix}(\alpha_p,\,k)\\(b_q,\,k)\end{matrix}\right.\right] = (k)^{-1}\,\overset{m,n}{\underset{p,q}{G}}\left[x^{1/k}\left|\begin{matrix}(\alpha_p)\\(b_q)\end{matrix}\right.\right],$$

k being a positive integer, we get

(1.7)
$$\int_0^\infty t^{2\nu-1}\varphi(t)\, H^{m',n'}_{p,q}\left[zt^{-2n}\,\middle|\,\begin{matrix}(a_p,k)\\(b_q,k)\end{matrix}\right]\mathrm{d}t =$$

$$= \left(k^{-1}(2\pi)^{(\frac{1}{2}-n)(n_1+n_2-m_1-m_2)}\right)\times\left((2n)^{\sum_1^{n_1}(a_q)+\sum_1^{n_2}(b_q)}\right)\times$$

$$\times\left((2n)^{-\sum_1^{m_1}(\delta_q)-\sum_1^{m_2}(\eta_q)+2\nu(n_1-n_2-m_1+m_2)-\frac{1}{2}(n_1+n_2-m_1-m_2)}\right)\times$$

$$\times\left(G^{m'+2nn_1,\,n'+2nn_2}_{P',\,Q'}\left[(2n)^{2n/k\,(n_2-n_1+m_1-m_2)}\,z^{1/k}\,\middle|\,I_2\right]\right),$$

where $P' = 2nn_2 + p + 2nm_1$, $Q' = 2nn_1 + q + 2nm_2$ and

(1.8)
$$I_2 = \begin{bmatrix}\Delta_{2n}[-2n;\,C],\ (a_p),\ \Delta(2n;\,C')\\ \Delta[2n;\,D],\ (b_q),\ \Delta_{2n}[-2n;\,D']\end{bmatrix}.$$

In (1.8), $C = (b_{n_2}) - 2\gamma - 2n$, $C' = (\delta_{m_1}) + 2\gamma$, $D = (a_{n_1}) + 2\gamma$ and $D' = (\eta_{m_2}) -$ $- 2\gamma - 2n$.

References.

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Abstract.

In this paper a generalisation of infinite integrals involving H-functions is established. This generalisation includes various integrals as particular cases, some of them are found by the author and others.

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