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Logarithmic proximate order and geometric means of an entire function of order zero. (**)

1. - Introduction.

For a non-constant entire function $f(z)$ of order zero, the L-order (logarithmic order), ϱ^*, and the lower L-order, λ^*, are given as [8]:

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r,f)}{\log\log r} = \begin{matrix}\varrho^*\\ \lambda^*\end{matrix} \qquad (1 \leqslant \lambda^* \leqslant \varrho^* \leqslant \infty),$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Let us define the following geometric means of $f(z)$ for $0 < k < \infty$,

$$G(r) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(r\exp(i\theta))|\, d\theta\right\}, \tag{1.1}$$

$$g_k(r) = \exp\left\{\frac{k+1}{r^{k+1}}\int_0^r x^k \log G(x)\,\mathrm{d}x\right\}, \tag{1.2}$$

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and

$$g_k^*(r) = \exp\left\{\frac{k+1}{(\log r)^{k+1}} \int_1^r (\log x)^k \log G(x) \frac{dx}{x}\right\}. \tag{1.3}$$

The mean value (1.2) was introduced by Kamthan [4] and a number of results regarding its growth with respect to $G(r)$ and other auxiliary functions for an entire function of order ϱ were obtained in [2], $[2]_1$, [4], [5]. In a recent paper [3], we have introduced a new geometric mean $g_k^*(r)$ as defined in (1.3), and various relations involving the comparative growths of $G(r)$, $g_k(r)$ and $g_k^*(r)$ relative to each other for an entire function of order zero have been established. It has been noted therein that the L-orders and the lower L-orders of the logarithms of these means are the same. Besides, the differences in the results regarding the growths of the pairs $(G(r), g_k(r))$ and $(G(r), g_k^*(r))$ have also been observed. The object of this paper is to continue a similar type of study by introducing L-proximate orders and thereby finding out the growth of $G(r)$ and $g_k^*(r)$. In section 2, we discuss certain preliminaries, whereas the remaining sections are devoted to our main results.

2. - Preliminaries.

It is assumed (throughout) that $f(z)$ is a non-constant entire function of order zero. For these functions, we have

$$\varrho_1 = \text{g.l.b.}\ \{\alpha\colon \alpha > 0 \ \text{and} \ \sum_{n=1}^{\infty} r_n^{-\alpha} < \infty\} = 0\,,$$

where $\{r_n\}_{n=1}^{\infty}$ denotes the sequence of the moduli of the zeros of $f(z)$. To have a more precise description of the distribution of the zeros of such functions, we define a number ϱ_1^* as

$$\varrho_1^* = \text{g.l.b.}\ \{\alpha\colon \alpha > 0 \ \text{and} \ \sum_{n=1}^{\infty} (\log r_n)^{-\alpha} < \infty\}\,,$$

and call it the L-convergence (logarithmic convergence) exponent of the zeros of $f(z)$ in analogy with ϱ_1 the convergence esponent of the zeros. Recently, the authors have proved that $[3]_1$:

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log\log r} = \varrho_1^* \qquad (0 \leqslant \varrho_1^* \leqslant \infty)\,, \tag{2.1}$$

where $n(r)$ is the number of the zeros of $f(z)$ in the disc $|z| \leqslant r$, and that $\varrho^* = \varrho_1^* + 1$. Also, we denote the limit inferior in (2.1) by λ_1^* and name it the lower L-convergence exponent of the zeros of $f(z)$ in analogy with λ_1, the lower convergence exponent of the zeros, i.e.

$$\liminf_{r\to\infty} \frac{\log n(r)}{\log\log r} = \lambda_1^* \qquad (0 \leqslant \lambda_1^* \leqslant \infty) . \tag{2.2}$$

Also let

$$N(r) = \int_0^r \frac{n(x)}{x}\, dx , \tag{2.3}$$

where it is assumed, without any loss of generality, that $n(r) = 0$ for $r \leqslant 1$.

We define $\mu(r)$, a real-valued function, to be a L-proximate (logarithmic proximate) order if it satisfies the following conditions:

(i) $\mu(r)$ is continuous and differentiable in adjacent intervals for $r \geqslant r_0$,

(ii) $\lim_{r\to\infty} \mu(r) = \mu \qquad (0 < \mu < \infty)$,

and

(iii) $\lim_{r\to\infty} r\cdot\mu'(r)\cdot\log r\cdot\log\log r = 0$,

where $\mu'(r)$ is either the right or the left-hand derivative at points where they are different.

We state below the existence theorem for the L-proximate order which can be easily proved on the lines of Levin ([**6**] p. 35):

Theorem A. If $F(r)$ is any function that is positive for $r > 1$ and satisfies the conditions:

$$\mu = \limsup_{r\to\infty} \frac{\log F(r)}{\log\log r} < \infty \qquad (\mu > 0) , \tag{2.4}$$

then L-proximate order $\mu(r)$ can be chosen so that

(iv) $F(r) \leqslant (\log r)^{\mu(r)}$

for $r \geqslant r_0$, and

(v) $F(r) = (\log r)^{\mu(r)}$

for a sequence r_n $(n = 1, 2, 3, \ldots)$ of values of r tending to infinity.

Further, if ν $(0<\nu<\infty)$ be the limit inferior in (2.4), then following the lines of Shah [7] it is easy to prove the existence of lower L-proximate order $\nu(r)$ having the following conditions:

(i)′ $\nu(r)$ is real, continuous and differentiable in adjacent intervals for $r\geqslant r_0$,

(ii)′ $\lim_{r\to\infty}\nu(r)=\nu$ $\qquad (0<\nu<\infty)$,

(iii)′ $\lim_{r\to\infty} r\cdot\nu'(r)\cdot\log r\cdot\log\log r=0$,

where $\nu'(r)$ is the right-hand or left-hand derivative where the two differ,

(iv)′ $F(r)\geqslant(\log r)^{\nu(r)}$, for $r\geqslant r_0$,

and

(v)′ $F(r)=(\log r)^{\nu(r)}$,

for a sequence r_m $(m=1,2,3,\ldots)$ of values of r tending to infinity.

Now, computing exactly on the lines of Levin ([**6**], pp 33-35) one can deduce that:

(a) $(\log r)^{\mu(r)}$ is a monotone increasing function of r, for $r\geqslant r_0$, $\mu>0$;

(b) for $r\to\infty$ and $0<a\leqslant k<b<\infty$, the asymptotic inequality

$$(1-\varepsilon)k^\mu(\log r)^{\mu(r)}<(\log r^k)^{\mu(r^k)}<(1+\varepsilon)k^\mu(\log r)^{\mu(r)}$$

holds uniformly in k;

(c) for $p<\mu+1$, $\displaystyle\int_{r_0}^{r}(\log t)^{\mu(t)-p}\frac{dt}{t}\sim\frac{(\log r)^{\mu(r)+1-p}}{(\mu+1-p)}$

and

(d) for $p>\mu+1$, $\displaystyle\int_{r}^{\infty}(\log t)^{\mu(t)-p}\frac{dt}{t}\sim\frac{(\log r)^{\mu(r)+1-p}}{(p-\mu-1)}$.

Also, following Singh and Dwivedi [**9**], we can easily obtain the various properties for lower L-proximate order $\nu(r)$ analogous to (a)-(d) of $\mu(r)$.

3. - Comparative growth of log $G(r)$ and log $g_k^*(r)$.

Theorem 3.1. *If $f(z)$ be an entire function of L-convergence exponent ϱ_1^* $(0<\varrho_1^*<\infty)$ and lower L-convergence exponent λ_1^* $(0<\lambda_1^*<\infty)$, then*

$$\liminf_{r\to\infty}\frac{\log g_k^*(r)}{\log G(r)}\leqslant\frac{k+1}{\varrho_1^*+k+2}, \tag{3.1}$$

and

$$\limsup_{r\to\infty}\frac{\log g_k^*(r)}{\log G(r)}\geqslant\frac{k+1}{\lambda_1^*+k+2}. \tag{3.2}$$

Proof. It is known that (see [**3**], theorem 1)

$$\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log G(r)}{\log\log r}=\frac{\varrho_1^*+1}{\lambda_1^*+1}. \tag{3.3}$$

Set $\varrho_1^*+1=\mu$ and $\lambda_1^*+1=\nu$. Since (3.3) is satisfied and $1<\nu,\ \mu<\infty$, there exist a L-proximate order $\mu(r)$ and a lower L-proximate order $\nu(r)$ for the function $\log G(r)$ satisfying the conditions (i)-(v) and (i)$'$-(v)$'$ respectively in section 2 where $F(r)$ is replaced by $\log G(r)$.

Now, from (1.3) we have

$$\log g_k^*(r)=\frac{k+1}{(\log r)^{k+1}}\int_1^r(\log x)^k\log G(x)\frac{dx}{x}\leqslant$$

$$\leqslant O\big((\log r)^{-k-1}\big)+\frac{k+1}{(\log r)^{k+1}}\int_{r_0}^r(\log x)^{\mu(x)+k}\frac{dx}{x}\sim\frac{(k+1)(\log r)^{\mu(r)}}{(\varrho_1^*+k+2)}\big(1+O(1)\big),$$

$$r\geqslant r_0=\frac{(k+1)\log G(r)}{(\varrho_1^*+k+2)}\big(1+O(1)\big),$$

for $r=r_n\to\infty$ as $n\to\infty$. Hence, (3.1) follows.

Similarly, making use of the lower L-proximate order $\nu(r)$ instead of $\mu(r)$, (3.2) is obtained.

4. - Growth of $\log G(r)$ and $\log g_k^*(r)$ relative to $n(r)$.

Theorem 4.1. *Under the hypothesis of theorem* 3.1, *we have*

$$\liminf_{r\to\infty} \frac{\log G(r)}{n(r)\log r} \leqslant \frac{1}{\varrho_1^* + 1}, \tag{4.1}$$

$$\limsup_{r\to\infty} \frac{\log G(r)}{n(r)\log r} \geqslant \frac{1}{\lambda_1^* + 1}, \tag{4.2}$$

$$\liminf_{r\to\infty} \frac{\log g_k^*(r)}{n(r)\log r} \leqslant \frac{k+1}{(\varrho_1^*+1)(\varrho_1^*+k+2)}, \tag{4.3}$$

and

$$\limsup_{r\to\infty} \frac{\log g_k^*(r)}{n(r)\log r} \geqslant \frac{k+1}{(\lambda_1^*+1)(\lambda_1^*+k+2)}. \tag{4.4}$$

Proof. In view of (2.1), (2.2) and $0<\lambda_1^*, \varrho_1^*<\infty$, there exist a *L*-proximate order $\varrho_1^*(r)$ and a lower L-proximate $\lambda_1^*(r)$ relative to $n(r)$ satisfying the conditions (i)-(v) and (i)'-(v)' respectively in section 2 where $F(r)$ is replace by $n(r)$.

Now, by Jensen's theorem (see Boas [**1**] p. 2)

$$\log G(r) = O(1) + \int_{r_0}^{r} \frac{n(x)}{x}\,dx \leqslant O(1) + \int_{r_0}^{r} (\log x)^{\varrho_1^*(x)} \frac{dx}{x} \sim \tag{4.5}$$

$$\sim \frac{(\log r)^{\varrho_1^*(r)+1}}{(\varrho_1^*+1)} \underset{(r\geqslant r_0)}{=} \frac{n(r)\log r}{\varrho_1^*+1},$$

for a sequence $r=r_n\to\infty$ as $n\to\infty$, so (4.1) is proved.

Similarly, making use of $\lambda_1^*(r)$ instead of $\varrho_1^*(r)$, (4.2) follows.

Further, for $1<r_0<r$, we have

$$\log g_k^*(r) = O\big((\log r)^{-k-1}\big) + \frac{k+1}{(\log r)^{k+1}} \int_{r_0}^{r} \log G(x)\ (\log x)^k \frac{dx}{x}.$$

Therefore, in view of (4.5), we see that

$$\log g_k^*(r) \leqslant O((\log r)^{-k-1}) + \frac{k+1}{(\log r)^{k+1}} \int_{r_0}^{r} \frac{(\log x)^{\varrho_1^*(x)+k+1}}{\varrho_1^*+1} \frac{dx}{x}$$

$$\sim \frac{(k+1)(\log r)^{\varrho_1^*(r)+1}}{(\varrho_1^*+1)(\varrho_1^*+k+2)}(1+O(1)) \underset{(r\geqslant r_0)}{=} \frac{(k+1)n(r)\log r}{(\varrho_1^*+1)(\varrho_1^*+k+2)}(1+O(1)),$$

for $r = r_m \to \infty$ as $m \to \infty$. Consequently, (4.3) is established. The inequality (4.4) can similarly be disposed on by using $\lambda_1^*(r)$ instead of $\varrho_1^*(r)$.

5. - Growth of log $G(r)$ and log $g_k^*(r)$ relative to an auxiliary function involving L-proximate order.

Let $f(z)$ be an entire function having L-order ϱ^* ($\varrho^* < \infty$) and L-proximate order $\varrho^*(r)$. Further, let

$$\varphi(r) = \frac{k+1}{(\log r)^{k+1}} \int_{1}^{r} (\log x)^k A(x)\, \varrho^*(x) \frac{dx}{x}, \qquad A(x) \equiv \log G(x).$$

Then $\varphi(r) \sim \varrho^* \log g_k^*(r)$ as $r \to \infty$. Define:

$$\lim_{r\to\infty} \frac{\sup}{\inf} \frac{\varphi(r)}{\psi(r)} = \frac{\alpha}{\beta}, \qquad \lim_{r\to\infty} \frac{\sup}{\inf} \frac{A(r)}{\psi(r)} = \frac{\gamma}{\delta},$$

where

$$\psi(r) = \exp \int_{c} \frac{\varrho^*(x)}{x \log x} dx, \qquad c > 1.$$

Now, we prove:

Theorem 5.1.

(5.1) $$\alpha \leqslant \frac{(k+1)\varrho^*\gamma}{(\varrho^*+k+1)},$$

(5.2) $$\beta \leqslant \varrho^* \delta \left(\frac{\delta}{\gamma}\right)^{(k+1)/\varrho^*} \left\{\left(\frac{\gamma}{\delta}\right)^{(k+1)/\varrho^*} - \frac{\varrho^*}{\varrho^*+k+1}\right\},$$

(5.3) $$\alpha \geqslant \frac{(k+1)\varrho^*\gamma}{(\varrho^*+k+1)} \left\{\frac{\varrho^*\gamma}{\gamma(\varrho^*+k+1)-\delta(k+1)}\right\}^{\varrho^*/(k+1)},$$

and

$$\beta \geqslant \frac{(k+1)\varrho^* \delta}{(\varrho^* + k + 1)}\,. \tag{5.4}$$

To prove this theorem, the following intermediate lemma is required:

Lemma. *For* $0 \leqslant \eta < \infty$:

$$\text{(i)} \int_{r_0}^{r} (\log x)^{k+1} \psi'(x)\, dx \sim \frac{\varrho^*}{(\varrho^* + k + 1)} (\log r)^{k+1} \psi(r)\,,$$

$$\text{(ii)} \int_{r}^{r^{1+\eta}} (\log x)^k \varrho^*(x) \frac{dx}{x} \sim \frac{\varrho^*}{k+1} (\log r)^{k+1}\big((1+\eta)^{k+1} - 1\big)\,,$$

and

$$\text{(iii)} \frac{\psi(r^{1+\eta})}{\psi(r)} \sim (1+\eta)^{\varrho^*}\,,$$

as $r \to \infty$.

Proof. We have

$$\frac{d}{dr}\big((\log r)^{k+1}\psi(r)\big) = (\log r)^{k+1} \psi'(r) \left\{1 + \frac{(k+1)}{r \log r} \frac{\psi(r)}{\psi'(r)}\right\} =$$

$$= (\log r)^{k+1} \psi'(r) \left(1 + \frac{k+1}{\varrho^*(r)}\right) \sim (\log r)^{k+1} \psi'(r) \left(\frac{\varrho^* + k + 1}{\varrho^*}\right),$$

as $r \to \infty$ and so (i) follows.

Since $\lim_{x\to\infty} \varrho^*(x) = \varrho^*$, one can easily see that

$$\int_{r}^{r^{1+\eta}} (\log x)^k \varrho^*(x) \frac{dx}{x} \sim \varrho^* \int_{r}^{r^{1+\eta}} (\log x)^k \frac{dx}{x}\,,$$

and

$$\log\left(\frac{\psi(r^{1+\eta})}{\psi(r)}\right) = \int_{r}^{r^{1+\eta}} \frac{\varrho^*(x)}{x \log x}\, dx \sim \varrho^* \log(1+\eta)\,,$$

as $r \to \infty$ Hence (ii) and (iii) are established.

Proof of the Theorem. Let $0 \leqslant \eta < \infty$ and $r_0 > 1$. Then

$$\varphi(r^{1+\eta}) = O\big((\log r)^{-k-1}\big) + \frac{(k+1)}{(1+\eta)^{k+1}(\log r)^{k+1}} \int_{r_0}^{r} (\log x)^k A(x)\, \varrho^*(x)\, \frac{dx}{x}$$

$$+ \frac{(k+1)}{(1+\eta)^{k+1}(\log r)^{k+1}} \int_{r}^{r^{1+\eta}} (\log x)^k A(x)\, \varrho^*(x)\, \frac{dx}{x}$$

$$= O\big((\log r)^{-k-1}\big) + \frac{(k+1)}{(1+\eta)^{k+1}(\log r)^{k+1}} \int_{r_0}^{r} (\log x)^{k+1} A(x)\, \frac{\psi'(x)}{\psi(x)}\, dx$$

$$+ \frac{(k+1)}{(1+\eta)^{k+1}(\log r)^{k+1}} \int_{r}^{r^{1+\eta}} (\log x)^k A(x)\, \varrho^*(x)\, \frac{dx}{x}$$

$$< O\big((\log r)^{-k-1}\big) + \frac{(k+1)(\gamma+\varepsilon)}{(1+\eta)^{k+1}(\log r)^{k+1}} \int_{r_0}^{r} (\log x)^{k+1} \psi'(x)\, dx$$

$$+ \frac{(k+1)A(r^{1+\eta})}{(1+\eta)^{k+1}(\log r)^{k+1}} \int_{r}^{r^{1+\eta}} (\log x)^k\, \varrho^*(x)\, \frac{dx}{x}$$

$$\sim \frac{(k+1)(\gamma+\varepsilon)\varrho^*}{(\varrho^*+k+1)(1+\eta)^{k+1}}\, \psi(r) + \varrho^* \left(1 - \frac{1}{(1+\eta)^{k+1}}\right) A(r^{1+\eta}),$$

using (i) and (ii) of the lemma. Therefore

$$\frac{\varphi(r^{1+\eta})}{\psi(r^{1+\eta})} < \frac{(k+1)(\gamma+\varepsilon)\varrho^*}{(\varrho^*+k+1)(1+\eta)^{k+1}}\; \frac{\psi(r)}{\psi(r^{1+\eta})} +$$

$$+ \varrho^* \left(1 - \frac{1}{(1+\eta)^{k+1}}\right) \frac{A(r^{1+\eta})}{\psi(r^{1+\eta})}.$$

Hence

$$\alpha \leqslant \frac{(k+1)\varrho^*\gamma}{(\varrho^*+k+1)(1+\eta)^{\varrho^*+k+1}} + \left(1 - \frac{1}{(1+\eta)^{k+1}}\right) \varrho^*\, \gamma\,,$$

and

$$\beta \leqslant \frac{(k+1)\,\varrho^* \gamma}{(\varrho^* + k + 1)(1+\eta)^{\varrho^* + k + 1}} + \left(1 - \frac{1}{(1+\eta)^{k+1}}\right) \varrho^* \delta \,.$$

Substituting the best values of η namely $\eta = 0$ and $\eta = (\gamma/\delta)^{1/\varrho^*} - 1$ in the above two inequalities, we get (5.1) and (5.2) respectively.

Similarly, we have

$$\frac{\varphi(r^{1+\eta})}{\psi(r^{1+\eta})} > \frac{(k+1)(\delta - \varepsilon)\,\varrho^*}{(\varrho^* + k + 1)(1+\eta)^{k+1}} \, \frac{\psi(r)}{\psi(r^{1+\eta})} + \varrho^* \left(1 - \frac{1}{(1+\eta)^{k+1}}\right) \frac{A(r)}{\psi(r)} \, \frac{\psi(r)}{\psi(r^{1+\eta})} \,.$$

Therefore

$$\alpha \geqslant \frac{(k+1)\,\varrho^* \delta}{(\varrho^* + k + 1)(1+\eta)^{\varrho^* + k + 1}} + \left(\frac{1}{(1+\eta)^{\varrho^*}} - \frac{1}{(1+\eta)^{\varrho^* + k + 1}}\right) \varrho^* \gamma \,,$$

and

$$\beta \geqslant \frac{(k+1)\,\varrho^* \delta}{(\varrho^* + k + 1)(1+\eta)^{\varrho^* + k + 1}} + \left(\frac{1}{(1+\eta)^{\varrho^*}} - \frac{1}{(1+\eta)^{\varrho^* + k + 1}}\right) \varrho^* \delta \,.$$

Substituting

$$\eta = \left(\frac{(\varrho^* + k + 1)\gamma - (k+1)\,\delta}{\varrho^* \gamma}\right)^{1/k+1} - 1$$

and $\eta = 0$ in the above inequalities, we obtain (5.3) and (5.4) respectively.

Corollary. *If* $\gamma = \delta$ *then it follows that*

$$\alpha = \beta = \frac{(k+1)\,\varrho^* \gamma}{(\varrho^* + k + 1)} \,. \tag{5.5}$$

The converse of the result (5.5) also holds good and we prove it in the following theorem.

Theorem 5.2. *If* α, β $(0 < \beta, \alpha < \infty)$ *and* γ, δ $(0 < \delta, \gamma < \infty)$ *be defined as above and if* $\alpha = \beta$, *then*

$$\gamma = \delta = \frac{(\varrho^* + k + 1)\,\alpha}{(k+1)\,\varrho^*} \,.$$

Proof. Let $0 \leqslant \eta < \infty$. Then

$$\varrho^*\big((1+\eta)^{k+1}-1\big)\big(1+O(1)\big)A(r) = \frac{k+1}{(\log r)^{k+1}} A(r) \int\limits_r^{r^{1+\eta}} (\log x)^k \varrho^*(x) \frac{\mathrm{d}x}{x} \leqslant$$

$$\leqslant \frac{k+1}{(\log r)^{k+1}} \int\limits_r^{r^{1+\eta}} (\log x)^k A(x) \varrho^*(x) \frac{\mathrm{d}x}{x} = \frac{k+1}{(\log r)^{k+1}} \Big(\int\limits_1^{r^{1+\eta}} - \int\limits_1^{r} \Big) (\log x)^k A(x) \varrho^*(x) \frac{\mathrm{d}x}{x}$$

$$= (1+\eta)^{k+1}\varphi(r^{1+\eta}) - \varphi(r)\,.$$

Since, for arbitrarily small $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$\alpha - \varepsilon < \frac{\varphi(r)}{\psi(r)} < \alpha + \varepsilon\,,$$

therefore

$$\varrho^*\big((1+\eta)^{k+1}-1\big)\big(1+O(1)\big)A(r) < (\alpha+\varepsilon)(1+\eta)^{k+1}\psi(r^{1+\eta}) - (\alpha-\varepsilon)\psi(r) \sim$$

$$\sim \psi(r)\big\{\big((1+\eta)^{\varrho^*+k+1}-1\big)\alpha + \big((1+\eta)^{\varrho^*+k+1}+1\big)\varepsilon\big\}\,.$$

Hence

$$\limsup_{r\to\infty} \leqslant \frac{A(r)}{\psi(r)} \leqslant \frac{\alpha\{(1+\eta)^{\varrho^*+k+1}-1\}}{\varrho^*\{(1+\eta)^{k+1}-1\}}\,.$$

But η is arbitrary and so making $\eta \to 0$, we find that

(5.6) $$\limsup_{r\to\infty} \frac{A(r)}{\psi(r)} \leqslant \frac{(\varrho^*+k+1)\alpha}{\varrho^*(k+1)}\,.$$

Similarly, by considering the inequality

$$\varrho^*\big(1-(1-\eta)^{k+1}\big)\big(1+O(1)\big)A(r) \geqslant \frac{k+1}{(\log r)^{k+1}} \int\limits_{r^{1-\eta}}^{r} (\log x)^k A(x) \varrho^*(x) \frac{\mathrm{d}x}{x}$$

one can readily see that

(5.7) $$\liminf_{r\to\infty} \frac{A(r)}{\psi(r)} \geqslant \frac{(\varrho^*+k+1)\alpha}{\varrho^*(k+1)}\,.$$

Hence the theorem follows from (5.6) and (5.7).

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Abstract.

Analogous to the properties of a proximate order and a lower proximate order for an entire function of order ϱ $(0 < \varrho < \infty)$ *and lower order* λ $(0 < \lambda < \infty)$, *properties of a L-proximate order and a lower L-proximate order for an entire function of order zero with L-order* ϱ^* $(\varrho^* < \infty)$ *and lower L-order* λ^* $(\lambda^* < \infty)$ *have been considered, and used to study the various growth relations of the geometric means* $G(r)$ *and* $g_k^*(r)$ *for an entire function of order zero.*

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