Riv. Mat. Univ. Parma (4) 2 (1976), 95-100

V. P. Gupta and S. K. Anand (*)

# On the means of an entire Dirichlet series of order (R) zero. (**)

## 1. - Introduction.

A Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \qquad s = \sigma + it,$$

where $0 = \lambda_0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \to \infty$ as $n \to \infty$, which we assume to be absolutely convergent everywhere in the complex plane $\mathscr{C}$, is bounded in any left strip and defines an entire function. The order of $f(s)$ is defined as:

$$\lim_{\sigma\to\infty} \operatorname{Sup} \frac{\log\log M(\sigma)}{\sigma} = \varrho \qquad (0 \leqslant \varrho \leqslant \infty),$$

where $M(\sigma) = \operatorname{Sup}\{|f(\sigma + it)| : -\infty < t < \infty\}$.

To have a more precise description of the growth relation for a class of entire Dirichlet series of order (R) zero, i.e. for which $\varrho = 0$, we use the notions of logarithmic order (R), $\varrho^*$, and the lower logarithmic order (R), $\lambda^*$, as given by (see [1]$_3$, [2])

$$(1.1) \qquad \lim_{\sigma\to\infty} \begin{matrix}\operatorname{Sup}\\ \operatorname{Inf}\end{matrix} \frac{\log\log M(\sigma)}{\log \sigma} = \begin{matrix}\varrho^*\\ \lambda^*\end{matrix} \qquad (1 \leqslant \lambda^* \leqslant \varrho^* \leqslant \infty).$$

---

(*) Indirizzo degli Autori: V. P. Gupta, M. M. H. College, Ghaziabad (U.P.), India; S. K. Anand, Faculty of Mathematics, University of Delhi, Delhi-110007, India,

(**) Research work has been supported by the University Grants Commision, India and the Ministry of Education, Govt. of India. – Ricevuto: 27-VI-1972.

Consider the following mean values of $|f(s)|$

$$I_\delta(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^\delta \, dt \qquad (0 < \delta < \infty)\,,$$

$$m_{\delta,k}(\sigma) = \exp\,(k\sigma)^{-1} \int_0^\sigma \exp\,(kx) I_\delta(x)\, dx \qquad (0 < k < \infty)\,.$$

Kamthan and Jain have obtained a number of growth relations regarding these means in ([**1**]$_1$, [**1**]$_2$, [**3**]$_2$) for entire Dirichlet series of order (R), $\varrho$ $(0 < \varrho < \infty)$. In this Note, our main object is to discuss certain properties of these means for functions of logarithmic order (R), $\varrho^*$, and lower logarithmic order (R), $\lambda^*$.

**2.** – Theorem 1. *Let $f(s)$ be an entire function represented by Dirichlet series of logarithmic order $\varrho^*$ and lower logarithmic order $\lambda^*$. Then, for $1 \leqslant \delta < \infty$,*

$$\text{(2.1)} \qquad \lim_{\sigma\to\infty} \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \frac{\log \left\{\dfrac{I_\delta^{(1)}(\sigma)}{I_\delta(\sigma)}\right\}}{\log \sigma} = \frac{\varrho^* - 1}{\lambda^* - 1} \qquad (1 \leqslant \lambda^* \leqslant \varrho^* < \infty)$$

*where $I_\delta^{(1)}(\sigma) = I_\delta(\sigma, f^{(1)})$.*

The proof of this theorem is based upon the following lemmas.

Lemma 1. *For $0 < \delta < \infty$,*

$$\lim_{\sigma\to\infty} \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \frac{\log\log I_\delta(\sigma)}{\log \sigma} = \frac{\varrho^*}{\lambda^*}\,.$$

Proof. For $\eta > 0$, we have (see [**2**])

$$\text{(2.2)} \qquad I_\delta(\sigma) < M(\sigma) < O(1) I_\delta(\sigma + \eta)$$

which, on using (1.1), proves the lemma.

Lemma 2 ([**3**]$_2$). *For $\sigma \geqslant \sigma_0$ and $\delta \geqslant 1$*

$$I_\delta^{(1)}(\sigma) \geqslant \frac{I_\delta(\sigma) \log I_\delta(\sigma)}{\sigma} \big(1 + O(1)\big)\,.$$

Lemma 3. ([**3**]$_2$). *With the usual notation for* $I_\delta^{(1)}(\sigma)$, *for all* $\sigma > 0$ *and* $\eta > 0$,

$$I_\delta^{(1)}(\sigma) \leqslant \frac{K}{\eta} I_\delta(\sigma + \eta) ,$$

*where* $K$ *is a constant.*

Proof of Theorem 1. Since $\log I_\delta(\sigma)$ is a convex function with respect to $\sigma$ (see [**3**]$_2$, lemma 5), we have

$$\log I_\delta(\sigma) = \log I_\delta(\sigma_0) + \int_{\sigma_0}^{\sigma} \omega(x)\, dx \qquad (\sigma > \sigma_0) , \tag{2.3}$$

where $\omega(x)$ is non-decreasing and almost continuous in the interval $(0, \infty)$; $w(x)$ tends to infinity with $x$. Therefore, for $\eta > 0$

$$\log I_\delta(\sigma + \eta) = \log I_\delta(\sigma) + \int_{\sigma}^{\sigma+\eta} w(x)\, dx \leqslant \log I_\delta(\sigma) + \eta w(\sigma + \eta)$$

which, on using Lemma 3, gives

$$\log I_\delta^{(1)}(\sigma) < \log I_\delta(\sigma) + \eta w(\sigma + \eta) - \log \eta + O(1) . \tag{2.4}$$

Choose $\eta = \big(w(\sigma + 2)\big)^{-1}$.

Then, $\eta w(\sigma + \eta) \leqslant 1$, for all sufficient great values of $\sigma$. Hence

$$\log I_\delta^{(1)}(\sigma) \leqslant \log I_\delta(\sigma) + \log w(\sigma + 2) + O(1) . \tag{2.5}$$

Also, from (2.3) and Lemma 1, it follows that

$$\lim_{\sigma \to \infty} \frac{\text{Sup}}{\text{Inf}} \frac{\log w(\sigma)}{\log \sigma} = \frac{\varrho^* - 1}{\lambda^* - 1} . \tag{2.6}$$

This, from (2.5) (2.6), we find that

$$\lim_{\sigma \to \infty} \frac{\text{Sup}}{\text{Inf}} \frac{\log \{I_\delta^{(1)}(\sigma)/I_\delta(\sigma)\}}{\log \sigma} \leqslant \frac{\varrho^* - 1}{\lambda^* - 1} .$$

The reverse inequality is easily available from Lemma 1 and 2.

Theorem 2. *Let $f(s)$ be an entire function represented by Dirichlet series of logarithmic order $\varrho^*$ and lower logarithmic order $\lambda^*$. Then for $\delta \geqslant 1$, $-1 < k < \infty$*

$$\lim_{\sigma\to\infty} \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \frac{\log \{m^{(1)}_{\delta,k}(\sigma)/m_{\delta,k}(\sigma)\}}{\log \sigma} = \begin{matrix}\varrho^*-1\\ \lambda^*-1\end{matrix},$$

*where $m^{(1)}_{\delta,k}(\sigma) = m_{\delta,k}(\sigma, f^{(1)})$.*

We omit the proof, as it can easily be followed on the lines of the proof of Theorem 1.

**3.** – It is known that, for all entire functions,

$$\lim_{\sigma\to\infty} \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \left\{\frac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)}\right\}^{1/\sigma} = \begin{matrix}e^\varrho\\ e^\lambda\end{matrix} \qquad (0 \leqslant \lambda \leqslant \varrho \leqslant \infty).$$

In particular, for entire functions of order (R) zero, i.e. $\varrho = 0$ we have

$$\lim_{\sigma\to\infty} \left\{\frac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)}\right\}^{1/\sigma} = 1\,. \tag{3.1}$$

In what follows, we have a result for entire functions of order (R) zero, which is more precise than (3.1), namely.

Theorem 3. *Let $f(s)$ be an entire function of logarithmic order $\varrho^*$ and lower logarithmic order $\lambda^*$. Then*

$$\lim_{\sigma\to\infty} \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \left\{\frac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)}\right\}^{1/\log\sigma} = \begin{matrix}\exp(\varrho^*-1)\\ \exp(\lambda^*-1)\end{matrix} \qquad (1 \leqslant \lambda^* \leqslant \varrho^* \leqslant \infty)\,.$$

Before proving this theorem, we will firstly prove the following

Lemma 4.

$$\lim_{\sigma\to\infty} \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \frac{\log\log m_{\delta,k}(\sigma)}{\log \sigma} = \begin{matrix}\varrho^*\\ \lambda^*\end{matrix}.$$

Proof. Lemma follows directly from Lemma 1 and the inequalities

$$m_{\delta,k}(\sigma) \leqslant \frac{I_\delta(\sigma)}{k} \leqslant m_{\delta,k}(\sigma+\eta)(1+o(1))^{-1} \qquad (\eta > 0)\,.$$

Proof of Theorem 3. It is seen, from the definition of $I_\delta(\sigma)$ and $m_{\delta,k}(\sigma)$, that (see [1]$_2$)

$$\log m_{\delta,k}(\sigma) = \log m_{\delta,k}(\sigma_0) + \int_{\sigma_0}^{\sigma} \nu^*(x)\,dx ,$$

where

$$\nu^*(x) = \left\{\frac{I_\delta(x)}{m_{\delta,k}(x)} - k\right\} \tag{3.2}$$

is an increasing function of $x$, for all large $x$ (see [3]$_2$, lemma 3). Thus, for all $\sigma \geqslant \sigma_0$

$$\log m_{\delta,k}(\sigma) - \log m_{\delta,k}(\sigma_0) \leqslant \nu^*(\sigma)(\sigma - \sigma_0) ,$$

which in view of Lemma 4, yields

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \nu^*(\sigma)}{\log \sigma} \geqslant \frac{\varrho^* - 1}{\lambda^* - 1} . \tag{3.3}$$

Again we have

$$\log m_{\delta,k}(2\sigma) \geqslant \int_{\sigma}^{2\sigma} \nu^*(x)\,dx \geqslant \sigma \nu^*(\sigma) ,$$

which again using Lemma 4, yields

$$\lim_{\sigma\to\infty} \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \frac{\log \nu^*(\sigma)}{\log \sigma} \leqslant \frac{\varrho^* - 1}{\lambda^* - 1} . \tag{3.4}$$

Hence, from (3.3) and (3.4), we get

$$\lim_{\sigma\to\infty} \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \frac{\log \nu^*(\sigma)}{\log \sigma} = \frac{\varrho^* - 1}{\lambda^* - 1} . \tag{3.5}$$

The theorem now follows from (3.2) and (3.5).

We wish to express our sincerest thanks and deep sense of gratitude to Dr. P. K. Jain for his most valuable guidance and constant help throughout the tenure of this work.

## References.

[1] P. K. Jain: $[\cdot]_1$ *Growth of the mean values of an entire functions represented by Dirichlet series*, Math. Nachr. **44** (1970), 91-97; $[\cdot]_2$ *Growth of the mean values of an entire function represented by Dirichlet series* (II), Rev. Roumaine Math. Pures Appl. **16** (1971), 1077-1084; $[\cdot]_3$ *Logarithmic order* (R) *and lower logarithmic order* (R) *of an entire Dirichlet series*, Fen. Fuk. Mecm. (to appear).

[2] P. K. Kamthan, *On the lower order* (R) *of entire Dirichlet series*, Math. Japan **9** (1964), 71-74.

[3] P. K. Kamthan and P. K. Jain: $[\cdot]_1$ *On entire function represented by Dirichlet series*, Rev. Roumaine Math. Pures Appl. **14** (1969), 361-366; $[\cdot]_2$ *On the mean values of an entire function represented by Dirichlet series*, Collect. Math. (to appear).

[4] J. F. Ritt, *On certain points in the theory of Dirichlet series*, Amer. J. Math. **50** (1926), 73-86.

## Summary.

*For an entire Dirichlet series* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ $(s = \sigma + i t, \lambda_1 \geq 0, \lambda_{n+1} \geq \lambda_n \to \infty$ *with* $n)$ *of order* (R) *zero, the logarithmic order* (R) $\varrho^*$ *and the lower logarithmic order* (R) $\lambda^*$ *have been defined as*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\log \sigma} = \begin{matrix}\varrho^*\\ \lambda^*\end{matrix} \qquad (1 \leqslant \lambda^* \leqslant \varrho^* \leqslant \infty),$$

*where* $M(\sigma) = \mathrm{Max}\,\{|f(\sigma + i\,t)| : -\infty < t < \infty\}$. *In this paper, certain proprietes of the mean values* $I_\delta(\sigma)$ *and* $m_{\delta,k}(\sigma)$ *of functions of logarithmic order* (R)$\varrho^*$ *and lower logarithmic order* (R)$\lambda^*$ *have been obtained.*

* * *