Riv. Mat. Univ. Parma (4) 2 (1976), 107-112

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On multiplication and weak multiplication modules. (**)

1. - Introduction.

In [2] the concept of multiplication module was introduced. In this paper we continue the study of multiplication module and also introduce the concept of a weak multiplication module, which is analogous to the concept of weak multiplication ring [3]. Although in general a weak multiplication ring is a multiplication ring [3], but in general, a weak multiplication module is not a multiplication module. In section 4, the examples of weak multiplication modules which are not multiplication modules are given. Theorem 1 of section 3 gives a characterization of multiplication module over a Dedekind domain in terms of projective module. It is also shown that if M is faithful along with the conditions of Theorem 1, then M is R-projective. In section 4, some results on weak multiplication modules are proved.

2. - Preliminaries.

All rings considered here are commutative which possess an identity element $1 \neq 0$ and all modules are unital left modules. A submodule N of a module M, which is not equal to M over a ring R is said to be *a prime submodule of* M if $AN_1 \subseteq N$ and $N_1 \not\subseteq N$ implies that $AM \subseteq N$, where A is an ideal of R and N_1 is a submodule of M. A module M over a ring R is called *a weak multiplication module* if $N \subseteq P$, where N is a submodule of M and P is a prime submodule of M, implies that there exists an ideal A of R such that $N = AP$.

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A ring R having a unique maximal ideal is called *a quasi-local ring*. A noetherian quasi-local ring is called *a local ring*. If S is any multiplicatively closed subset of a ring R, then M_S denotes the quotient module of M with respect to S. For each R-submodule N of M, N^e denotes the extension of N in M_S and for R_S-submodule L of M_S, L^c denotes the contraction of L in M. If R is a domain with field of quotient K and M is an R-module, then the rank of M is the dimension of $K \otimes_R M$ as a vector space over K, i.e., the rank of M is the maximum rank of free submodules of M. Q and Z are the set of rational numbers and integers respectively. On the whole terminology is of [**5**]$_1$, [**5**]$_2$ and [**1**].

3. - Multiplication modules.

Proposition 1. *If M is a multiplication R-module. Then a submodule N of M is prime if and only if $(N:M)$ is prime ideal of R.*

Proof. In general, for any prime submodule N of M, $(N:M)$ is prime ideal of R. Conversely, suppose that $(N:M)$ is prime ideal of R. Take $AL \subseteq N$ such that $L \not\subseteq N$, where L is submodule of M and A is an ideal of R. As M is multiplication module $L = BM$ for some ideal B of R. Then $AL = ABM \subseteq N$ i.e. $AB \subseteq (N:M)$, but $L = BM \not\subseteq N$ i.e., $B \not\subseteq (N:M)$. Therefore $A \subseteq (N:M)$, as $(N:M)$ is prime. Hence $AM \subseteq N$. This shows N is prime submodule of M. This completes the proof.

Converse is not true in general, is clear from the following

Example. Consider any domain D which is not a field. Consider $M = M_1 \oplus M_2$ such that $M_1 \cong M_2 \cong D$. Let A be a proper ideal of D which is not prime ideal and let A_2 be the corresponding submodule of M_2. Then there exists $x \in M_2$ and $a \in D$ such that $x \notin A_2$, $a \notin A$ but $ax \in A_2$. Now $aM \not\subseteq A_2$. Hence A_2 is not a prime submodule of M. However, $a \in (A_2:M)$ implies that $aM_1 \oplus aM_2 \subseteq A_2 \Rightarrow aM_1 = 0 \Rightarrow a = 0$ so that $(A_2:M) = (0)$, a prime ideal of D.

Theorem 1. *If M is a multiplication module over a Dedekind domain R, then either M is direct sum of cyclic R-modules or M is R-projective.*

Proof. Steinitz in [**4**], proved that every finitely generated module over a Dedekind domain is direct sum of cyclic modules and finitely generated torsion free modules of rank one. That is $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where some M_i are cyclic and some are torsion free of rank one. As M is multiplication module over a Dedekind domain, by lemma 3 [2], M is finitely generated and hence by theorem 3 [2]: $R = \operatorname{ann}(M_i) + \bigcap_{j \neq i}^{n} \operatorname{ann}(M_j)$. If M_i is torsion free, then $\operatorname{ann}(M_i) = 0$ and therefore $\bigcap_{j \neq i}^{n} \operatorname{ann}(M_j) = R$, i.e., $M_j = 0$ for $\forall j \neq i$. Hence $M = M_i$. Therefore M is torsion free module of rank one. But over a domain a finitely generated torsion free module of rank one is isomorphic to an ideal I of R. Now as R is a Dedekind domain every ideal of R is R-projective. Therefore M is also R-projective. Hence either M is direct sum of cyclic modules or M is R-projective. This completes the proof of the theorem.

Corollary. *If M is faithful along with the conditions of Theorem 1, then M is R-projective.*

Proof. Suppose N is torsion submodule of M. Then $N = AM$ for some ideal A of R. Suppose $aN = 0$ for $a(\neq 0) \in R$. Then $aAM = 0$, but M is faithful, therefore $aA = 0$. As $a \neq 0$, $A = 0$ and hence $N = 0$. Therefore M is torsion free. Now by above theorem M is torsion free module of rank one and hence R-projective. This completes the proof.

4. - Weak Multiplication Module.

The following proposition is immediate from the definition of weak multiplication module.

Proposition 1. *The homomorphic image of a weak multiplication module is weak multiplication module.*

Proposition 2. *If M is a weak multiplication R-module and S is a multiplicatively closed subset of R, then the quotient module M_S is weak multiplication R_S-module.*

Proof. Consider any two R_S-submodules N, L of M_S such that $N \subseteq L$ and L is prime. Then in M, $N^c \subset L^c$. We claim that L^c is prime submodule of M. Suppose $AK \subseteq L^c$ for some ideal A of R and some submodule K of M such that $K \nsubseteq L^c$. Now as $AK \subseteq L^c$, then $A^e K^e \subseteq L$, but $K^e \nsubseteq L$, therefore $A^e M_S \subseteq L$, hence $A^{ec} M_S^c \subseteq L^c$, i.e., $AM \subseteq L^c$. Therefore L^c is a prime submodule of M. As M is weak multiplication module, there exists an ideal A of R such that $N^c = AL^c$, then $N^{ce} = A^e L^{ce}$. Therefore $N = A^e L$. Hence M_S is R_S-weak multiplication module.

Proposition 3. *If M is a weak multiplication module over a noetherian (artinian) ring R, then the class of submodules of M which are contained in a prime submodule (which contains a prime submodule) of M satisfies maximal (minimal) condition.*

Proof. Let P be a prime submodule of a weak multiplication module M. Since by definition all submodules of P are of the form AP, where A is an ideal of R, the result follows.

Proposition 4. *If M is a module over a ring $R = S \oplus T$ with $1 = e + f$ (1 e and f identities of R, S and T respectively). Then a submodule P of M is prime submodule of M iff e P is prime submodule of eM and $fP = fM$ or fP is a prime submodule of fM and $eP = eM$.*

Proof. Suppose P is a prime submodule of M and $eP \neq eM$. Now $T(eM) = 0$, therefore $T(eM) \subseteq P$, as $eM \not\subseteq P$, $TM \subseteq P$. But $TM = fM$; therefore $fM \subseteq P$, hence $fM = fP$. To show eP in prime S-submodule of eM. Suppose $S_1 M_1 \subseteq eP$, where S_1 is an ideal of S and M_1 is a submodule of eM. If $M_1 \not\subseteq eP$ then $M_1 \not\subseteq eP \oplus fM = P$. Therefore $S_1 e M \subseteq P$ and hence $S_1 eM \subseteq eP$. This shows that eP is a prime submodule of eM, similarly if $fM \neq fP$, then fP is a prime submodule of fM. To show $P = eM \oplus fP$ is prime. Take $AN \subseteq P$. If $N \not\subseteq P$, then $fN \not\subseteq fP$, but $AN \subseteq P \Rightarrow AfN \subseteq fP$, therefore $AfM \subseteq fP$. Hence $AM \subseteq p$. Hence P is a prime submodule of M.

Proposition 5. *If M is a weak multiplication module and P and P' are prime submodules of M such that P is strictly contained in P'. Then $P = (P:M)P'$.*

Proof. By definition $P = AP'$ for some ideal A of R. Now $P' \not\subseteq P$ implies that $AM \subseteq P$, i.e. $A \subseteq (P:M)$. Now $P = AP' \subseteq (P:M)P' \subseteq P$. Therefore $P = (P:M)P'$. This completes the proof.

Proposition 6. *Let M be a weak multiplication module, P be a prime submodule of M. Then there is no submodule strictly between P and AP for any maximal idal A of R.*

Proof. Trivial.

Proposition 7. *Let M be a weak multiplication module over a quasi-local ring R, then any prime submodule N of M is cyclic.*

Proof. Let P be the maximal ideal of R. Suppose that $N = PN$. Consider any $x(\neq 0) \in N$. Then $Rx = AN$ for some ideal A of R. Then $P(x) = PAN = AN = Rx$. This implies that $x = px$ for some $p \in P$. Thus $x = 0$, as $(1-p)$ is unit. This is a contradiction, hence $N \neq PN$. Choose $x \in N \setminus PN$. Then $Rx = AN$. Now either $A = R$ or $A \subseteq P$. If $A \subseteq P$, then $Rx = AN \subseteq PN$, a contradiction. Therefore $Rx = RN = N$. Hence N is cyclic R-module. This completes the proof.

Proposition 8. *Any divisible uniform module over a domain R is weak multiplication module.*

Proof. Suppose N is a prime submodule of a divisible uniform module M. Take $x \in M \setminus N$. Then there exists a nonzero $t \in R$ such that $tx \in N$. Now $Rtx \subseteq N$. But $Rx \nsubseteq N$ implies that $tM \subseteq N$, but $tM = M \nsubseteq N$. Therefore $N = 0$. Hence M is a weak multiplication module. This completes the proof.

Now we give the examples of weak multiplication modules which are not multiplication modules.

Example 1. Q_Z is a weak multiplication module, but not a multiplication module.

If N is a prime submodule of Q. Take $x \in Q \setminus N$. As Q_Z is divisible there exists an integer $a(\neq 0) \in Z$ such that $xa \in N$. Then $Rxa \subseteq N$ but $Rx \nsubseteq N \Rightarrow Qa \subseteq N$ but $Qa = Q \nsubseteq N$. Therefore $N = 0$. Hence Q_Z is weak multiplication module. However, it is not a multiplication module.

Example 2. Let R be a local discrete valuation ring of rank one with maximal ideal M. Consider $N = R/M \oplus R$ as R-module. Then N is weak multiplication R-module, while N is not a multiplication module.

Clearly R is prime submodule of N. Let P be a prime submodule of N. If $P = 0$, then $(R/M)M = (0) \subseteq P$, but $NM = M \neq (0)$. So, (0) is not a prime submodule. Now $(R/M)M = (0) \subseteq P$ gives either $R/M \subseteq P$ or $NM \subseteq P$, i.e., $M \subseteq P$.

If $R/M \subseteq P$, then $P = R/M \oplus (P \cap R) \Rightarrow N/P \cong R/P \cap R \Rightarrow P \cap R$ is a prime ideal of R. So that $P = R/M \oplus P_1$, P_1 a prime ideal of R. Suppose now $M \subseteq P$ and $(R/M) \cap P = 0$. Now $p \in P$ implies that $p = \bar{r}_1 + r_2$, $\bar{r}_1 \in R/M$ and $r_2 \in R$. Define a mapping $\sigma\colon P \to R$ as $\sigma(p) = r_2$. Then clearly σ is an R-homomorphism with kernel zero, as $(R/M) \cap P = 0$. Therefore σ is a isomorphism. Now either $P \cong R$ or $P \cong M$. If $P \cong R$ then $\forall T \subseteq P$, $T = PA$ for some ideal A of R. Further as R is discrete valuation ring of rank one, M is a multiplication module, P is also multiplication module. Therefore N is a weak multiplication module. N is not a multiplication module, because if we take R/M, then $R/M \neq NT$, because $NT = (R/M)T \oplus RT = (R/M)T \oplus T$. If $T = R$, then $NT = N$. If $T \subseteq M$ then $(R/M)T = 0$ gives us $NT = T$ with $T \cap R/M = 0$. Therefore $R/M \neq NT$ for any ideal T.

The author takes the opportunity for thanking Dr. Surjeet Singh (Reader, Department of Mathematics and Statistics, Aligarh Muslim University) for his valuable suggestions in the preparation of this paper.

References.

[1] M. D. LARSEN and P. J. MC CARTHY, *Multiplication theory of ideals*, Academic Press, New York 1971.

[2] F. MEHDI, *On multiplication modules*, Maths. Student **42** (1974), 149-153.

[3] J. L. MOTT, *Equivalent conditions for a ring to be multiplication ring*, Canad. J. Math. **15** (1954), 429-434.

[4] E. STEINITZ, *Rechtliebige systeme und moduln in algebraischen zahlkörpern*, Math. Ann. **71** (1912), 328-354.

[5] O. ZARISKI and P. SAMUEL: $[\cdot]_1$ *Commutative Algebra* (I), D. Van Nostrand, New York 1958; $[\cdot]_2$ *Commutative Algebra* (II), D. Van Nostrand, New York 1960.

Abstract.

A module M over a commutative ring R is called a multiplication module if for every pair of submodules, N and L, $N \subseteq L$ implies $N = AL$ for some ideal A of R. A proper submodule P of a module M is said to be a prime submodule of M if $AN \subseteq P$ and $N \not\subseteq P$ implies that $AM \subseteq P$, where A is an ideal of R and N is a submodule of M. A module M is said to be weak multiplication module if $N \subseteq P$, where N is a submodule of M and P is a prime submodule of M, implies $N = AP$ for some ideal A of R. Following are some of the main results: (i) *If M is a multiplication module over a Dedekind domain R, then either M is direct sum of cyclic R-modules or M is R-projective.* (ii) *Any prime submodule of a weak multiplication module over a quasi-local ring is cyclic.* (iii) *Any divisible uniform module over a domain is weak multiplication module.*

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