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B. C. CHAKRABORTY (*)

On the algebraic and topological structure of a class of entire Dirichlet series. (**)

1. - Introduction.

Let Ω be the class of entire Dirichlet series having the same sequence, $\{\lambda_n\}$, of exponents. Let

(1.1) $$f(s) = \sum_{n=1}^{\infty} a_n \exp[s\lambda_n],$$

(1.2) $$g(s) = \sum_{n=1}^{\infty} b_n \exp[s\lambda_n] \qquad (s = \sigma + i\tau),$$

be two elements of Ω. Here we tacitly assume that $0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$. We now define two compositions, namely, addition (+) and star-multiplication (*) in the set Ω in the following manner:

$$f(s) + g(s) = \sum_{n=1}^{\infty} (a_n + b_n) \exp[s\lambda_n],$$

$$f(s) * g(s) = \sum_{n=1}^{\infty} a_n b_n \exp[s\lambda_n].$$

The object of this paper is to study the algebraic and topological structure of Ω. In this paper we shall follow Rudin's [1] definition of Banach algebra and not of Simmons' [2].

(*) Indirizzo: Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Calcutta 19, India.
(**) Ricevuto: 15-I-1974.

2. – Theorem 1. *The set Ω is closed with respect to the two compositions $+$ and $*$.*

Proof. Let $f(s)$, $g(s) \in \Omega$. Since both $f(s)$ and $g(s)$ are entire, they are bounded in $\sigma \leqslant x$ for any $x < \infty$ and the series

$$\sum_{n=1}^{\infty} |a_n| \exp[\sigma\lambda_n]\,, \qquad \sum_{n=1}^{\infty} |b_n| \exp[\sigma\lambda_n]$$

are convergent for all σ. Hence, $\sum_{n=1}^{\infty}\{|a_n| + |b_n|\} \exp[\sigma\lambda_n]$ is convergent for all σ and consequently $\sum_{n=1}^{\infty} |a_n + b_n| \exp[\sigma\lambda_n]$ is convergent for all σ. Hence, $f(s) + g(s)$ is an entire function. Again, due to the convergence of $\sum_{n=1}^{\infty} |b_n| \exp[\sigma\lambda_n]$ for all σ, it follows that $\lim_{n\to\infty} |b_n| = 0$ and hence $\sum_{n=1}^{\infty} |a_n|\,|b_n| \exp[\sigma\lambda_n]$ i.e., $\sum_{n=1}^{\infty} |a_n b_n| \exp[\sigma\lambda_n]$ is convergent for all σ, which implies that $f(s) * g(s)$ is an entire function.

Theorem 2. *Ω is an additive abelian group.*

Proof. Let $h(s) = \sum_{n=1}^{\infty} c_n \exp[s\lambda_n] \in \Omega$. Evidently, $f(s) + g(s) = g(s) + f(s)$ and $f(s) + [g(s) + h(s)] = [f(s) + g(s)] + h(s)$; $\varphi(s) = \sum_{n=1}^{\infty} a_n \exp[s\lambda_n]$, where $a_n = 0$ for all n, is the null element of Ω. Since $\sum_{n=1}^{\infty} a_n \exp[s\lambda_n]$ is entire implies $\sum_{n=1}^{\infty}(-a_n) \exp[s\lambda_n]$ is entire, the negative element of

$$f(s) = \sum_{n=1}^{\infty} a_n \exp[s\lambda_n] \qquad \text{is} \qquad \sum_{n=1}^{\infty}(-a_n) \exp[s\lambda_n] \in \Omega\,.$$

Hence, using Theorem 1, Ω is an additive abelian group.

Theorem 3. *Ω is a normed complex linear space.*

Proof. First we show that Ω is a linear space over the field c of complex numbers. We define scalar multiplication as follows: $x\cdot f(s) = \sum_{n=1}^{\infty} x a_n \exp[s\lambda_n]$, $x \in c$. Evidently, $x\cdot f(s) \in \Omega$.

We can easily verify the following:

$$\begin{aligned}
&\text{(i)} && 1\cdot f(s) &&= f(s),\\
&\text{(ii)} && (x+y)\cdot f(s) &&= x\cdot f(s) + y\cdot f(s),\\
&\text{(iii)} && (xy)\cdot f(s) &&= x\{y\cdot f(s)\},\\
&\text{(iv)} && x\cdot\{f(s)+g(s)\} &&= x\cdot f(s) + x\cdot g(s),
\end{aligned}$$

where $x, y \in c$ and $f(s)$, $g(s) \in \Omega$.
Hence, using Theorem 2, Ω is a complex linear space. Now, we define the norm of $f(s)$ by $\|f(s)\| = \sup_n |a_n|$. We observe that

(i) $\|f(s)\| \geqslant 0$ and $\|f(s)\| = 0 \Leftrightarrow f(s) = 0$;

(ii) $\|f(s)+g(s)\| = \sup_n |a_n + b_n| \leqslant \sup_n |a_n| + \sup_n |b_n| = \|f(s)\| + \|g(s)\|$;

(iii) $\|x\cdot f(s)\| = \sup_n |xa_n| = |x| \sup_n |a_n| = |x|\,\|f(s)\|$.

Hence, Ω is a normed complex linear space.

Remark. We observe that Ω is a metric space with respect to the metric d defined by $d(f, g) = \|f(s) - g(s)\|$. If $f \in \Omega$ and $r > 0$, the open ball with centre at f and radius r is the set $\{g \in \Omega\colon d(f, g) < r\}$. Now, if Λ is the collection of all sets $E \subset \Omega$ which are arbitrary unions of open balls, then Λ is a topology in Ω and Ω becomes a topological space.

Theorem 4. *Ω is a complex Banach space.*

Proof. Let the sequence $\{f_p(s)\}$, where $f_p(s) = \sum_{n=1}^{\infty} a_{p,n} \exp[s\lambda_n] \in \Omega$ for $p = 1, 2, \ldots$, be a Cauchy sequence. Then for every $\varepsilon > 0$ there exists a positive integer m such that $\|f_p(s) - f_q(s)\| < \varepsilon$ for $p, q \geqslant m$. Hence, $\sup_n |a_{p,n} - a_{q,n}| < \varepsilon$ for $p, q \geqslant m$. This implies

$$|a_{p,n} - a_{q,n}| < \varepsilon \quad \text{for } p, q \geqslant m \text{ and for all } n. \tag{1.3}$$

We fix n and consider the sequence $a_{1,n}, a_{2,n}, \ldots$. Due to (1.3) this sequence will converge to a limit, say, a_n. We now form the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n \exp[s\lambda_n]$. From (1.3) we have

$$|a_{p,n} - a_n| < \varepsilon \quad \text{for } p \geqslant m \text{ and for all } n. \tag{1.4}$$

Hence

$$(1.5) \qquad |a_n| < |a_{m,n}| + \varepsilon \qquad \text{for all } n\,.$$

But $\sum_{n=1}^{\infty} a_{m,n} \exp[s\lambda_n] \in \Omega$ implies $\sum_{n=1}^{\infty} |a_{m,n}| \exp[\sigma\lambda_n]$ is convergent for all $\sigma < \infty$. Hence, from (1.5) it follows that $\sum_{n=1}^{\infty} |a_n| \exp[\sigma\lambda_n]$ is convergent for all $\sigma < \infty$, i.e., $f(s)$ is an entire function and hence belongs to Ω. From (1.4), $\operatorname{Sup}_n |a_{p,n} - a_n| < \varepsilon$ for $p \geqslant m$; i.e., $\|f_p(s) - f(s)\| < \varepsilon$ for $p \geqslant m$. Hence, $f_p(s) \to f(s) \in \Omega$ when $p \to \infty$. Hence, using Theorem 3, Ω is a complex Banach space.

Lemma 1. *Ω is a commutative ring.*

Proof. We can easily verify the following properties in Ω:

(i) $f(s) * \{g(s) * h(s)\} = \{f(s) * g(s)\} * h(s)$,

(ii) $f(s) * g(s) = g(s) * f(s)$,

(iii) $f(s) * \{g(s) + h(s)\} = f(s) * g(s) + f(s) * h(s)$,

(iv) $\{f(s) + g(s)\} * h(s) = f(s) * h(s) + g(s) * h(s)$.

Hence using Theorems 1, 2 Ω is a commutative ring.

Lemma 2. *Ω is a commutative algebra over the field of complex numbers.*

Proof:

$$x\{f(s) * g(s)\} =$$

$$= \sum_{n=1}^{\infty} x a_n b_n \exp[s\lambda_n] = \sum_{n=1}^{\infty} x a_n \exp[s\lambda_n] * \sum_{n=1}^{\infty} b_n \exp[s\lambda_n] = \{xf(s)\} * g(s)\,.$$

Also, $\sum_{n=1}^{\infty} x a_n b_n \exp[s\lambda_n] = \sum_{n=1}^{\infty} a_n \exp[s\lambda_n] * \sum_{n=1}^{\infty} x b_n \exp[s\lambda_n] = f(s) * \{xg(s)\}$. Hence, $x\{f(s) * g(s)\} = \{x f(s)\} * g(s) = f(s) * \{xg(s)\}$ for all $f, g \in \Omega$, $x \in c$. Hence, by Theorem 3 and Lemma 1, Ω is a commutative algebra over c.

Theorem 5. *Ω is a commutaitve Banach algebra.*

Proof. $\|f(s) * g(s)\| = \sup_n |a_n b_n| \leqslant \sup_n |a_n| \sup_n |b_n| = \|f(s)\| \|g(s)\|$. Hence, by Theorem 4 and Lemma 2, Ω is a commutative Banach algebra.

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References.

[1] W. RUDIN, *Real and complex analysis*, Mc Graw-Hill, New York 1966.

[2] G. F. SIMMONS, *Topology and modern analysis*, Mc Graw-Hill, New York 1963.

Summary.

Let Ω be the class of entire Dirichlet series having the same sequence of exponents. We define two compositions, addition and starmultiplication in Ω in a suitable manner. We then show that Ω is a commutative Banach algebra.

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