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On unsteady non-Newtonian flow in a rotating system in the presence of a uniform magnetic field. (**)

1. - Introduction.

Debnath $[1]_1$, $[1]_2$ has recently initiated the study of an unsteady hydromagnetic boundary layer flow in a Newtonian rotating fluid in the presence of a uniform magnetic field. The configurations involved in these problems were (i) a semi-infinite expanse of Newtonian fluid bounded by an infinite rigid insulating disk and (ii) a laterally unbounded fluid between two infinite parallel insulating disks, when both the fluid and the disk (s) are in rigid body rotation with a constant angular velocity. The unsteady flow was generated in both the configurations by imposing non-torsional or torsional oscillations of the disk (s) with a given frequency. The exact solutions of the problems have been determined explicitly and the structure of the associated hydrodynamic and hydromagnetic boundary layers were investigated in detail. The analysis revealed the existence of Ekman-Hartmann and Stokes-Ekman-Hartmann boundary layers on the disk (s). For the problem of torsional oscillations of a disk in the Newtonian fluid, the total time scale involved for the establishment of quasi-steady boundary layers and the hydromagnetic Ekman suction velocity were exactly calculated. The significances of these results for the attainment of the hydromagnetic boundary layers and for the hydromagnetic spin-up mechanism were discussed. It has been shown that

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the interaction of the electromagnetic and Coriolis forces, and the Ekman suction velocity have some distinctive role on the structure of the boundary layers and the hydromagnetic spin-up mechanism.

Based on the Walters rheological equations of state for a class of non-Newtonian fluid, Basu and Debnath [2] have investigated the unsteady problem of non-torsional oscillations of a disk in an incompressible, homogeneous visco-elastic fluid in a rotating coordinate system. The unsteady velocity field for configuration (i) has been calculated explicitly, and the structure of the associated boundary layers has been determined. The analysis has provided the existence of Stokes-Ekman-Elastic boundary layers on the disk. It was shown that in the absence of the elastic parameter, the results of the work are in accord with the corresponding results of an uniformly rotating Newtonian fluid.

It appears that the flow generated in a semi-infinite electrically conducting visco-elastic fluid in the presence of a uniform magnetic field by the non-torsional oscillations of an infinite disk may be of considerable interest in geophysical and rheological problems. It is thus natural to initiate some theoretical work with simple geometrical configurations before it can be applied and extended to problems of geophysical and rheological interest and to a more general physical configuration.

The present paper is based upon the rheological equations of state for a class of elastico-viscous fluid B and is essentially concerned with the study of the unsteady hydromagnetic boundary layer flow in an electrically conducting elastico-viscous fluid in the presence of a uniform magnetic field. A semi-infinite fluid is bounded by an infinite plate and the unsteady flow is generated by the non-torsional oscillations of the plate. The initial value problem is solved by the operational calculus of Heaviside.

This analysis is aimed at finding some qualitative and quantitative information about (i) the unsteady velocity distribution, (ii) the existence and structure of the associated Stokes-Ekman-Hartmann-elastic boundary layers on the plate and (iii) the significant interaction of the Coriolis, hydromagnetic and elastic parameters. It is shown that there exists a combined Stokes-Ekman-Hartmann-elastic boundary layers which remain bounded for both resonant and non-resonant cases. Attention is given to the behaviour of the transient solution for sufficiently large t. The surface traction at the plate is calculated in a closed form and its general features are discussed. Several particular results are recovered as special cases of this analysis.

2. - The constitutive equations of state, motion and continuity.

Walters $[3]_1$ formulated the rheological equations of state for the visco-elastic liquid B' which are

$$p_{ik} = -p g_{ik} + p'_{ik} \,, \tag{2.1}$$

$$p'_{ik}(x, t) = 2\int_{-\infty}^{t} \Psi(t-t') \frac{\partial x^i}{\partial x'^m} \frac{\partial x^k}{\partial x'^m} \exp[(1)mr](x', t')\, dt' \,, \tag{2.2}$$

where p_{ik} is the stress tensor, p is an isotropic pressure, g_{ik} is the metric tensor of fixed coordinate system x^i, p'_{ik} is the reduced stress tensor, x'^i is the position at time t' of the element which is instantaneously at the point x^i at time t, $e^{(1)}_{ik}$ is the rate of strain tensor and

$$\Psi(t-t') = \int_0^{\infty} \frac{N(\tau)}{\tau} \exp\left[-\frac{(t-t')}{\tau}\right] d\tau \,, \tag{2.3}$$

$N(\tau)$ being the distribution function of relaxation time τ.

The Oldroyd rheological equations of state [**4**] for liquid B

$$\left(p'^{ik} + \lambda_1 \frac{\delta}{\delta t} p'^{ik}\right) = 2\eta_0 \left(\exp[(1)ik] + \lambda_2 \frac{\delta}{\delta t} \exp[(1)ik]\right) \tag{2.4}$$

is a special case of the Walters liquid B' when

$$N(\tau) = \eta_0 \left(\frac{\lambda_2}{\lambda_1}\right) \delta(\tau) + \eta_0 \left(\frac{\lambda_1}{\lambda_2} - 1\right) \delta(\tau - \lambda_1) \tag{2.5}$$

is substituted in equations (2.2) and (2.3), where $\delta(\tau)$ is the Dirac function of time τ, η_0 is the coefficient of viscosity, λ_1, λ_2 $(< \lambda_1)$ are the relaxation and the retardation times respectively, and $\delta/\delta t$ represents convective derivative of a tensor quantity in relation to the material in motion. The convective derivative of any contravariant tensor a^{ik} is

$$\frac{\delta}{\delta t} a^{ik} = \frac{\partial}{\partial t} a^{ik} + v^m \frac{\partial a^{ik}}{\partial x^m} - \frac{\partial v^k}{\partial x^m} a^{im} - \frac{\partial v^i}{\partial x^m} a^{mk} \,, \tag{2.6}$$

where v^i is the velocity vector.

The constitutive equations for the Newtonian liquid and the Maxwell liquid follow from (2.4) when $\lambda_1 = \lambda_2$ and $\lambda_2 = 0$ respectively. The Newtonian liquid is also given by $N(\tau) = \eta_0\, \delta(\tau)$.

Walters $[\mathbf{3}]_2$ has also proved that for liquids of short memory (that is, short relaxation times), the equation of state has the simplified form

$$p'^{ik} = 2\eta_0 \exp[(1)\,ik] - 2k_0 \frac{\delta}{\delta t} \exp[(1)\,ik]\,, \tag{2.7}$$

where $\eta_0 = \int_0^\infty N(\tau)\,d\tau$ is the limiting viscosity at small rates of shear and $k_0 = \int_0^\infty \tau N(\tau)\,d\tau$.

Equations (2.1) and (2.7) constitute the equations of state for a class of liquids called « the Walters liquid B'' ».

For Oldroyd's liquid B, $k_0 = \eta_0(\lambda_1 - \lambda_2)$ so that (2.7) reduces to

$$p'^{ik} = 2\eta_0 \exp[(1)\,ik] - 2\eta_0\,(\lambda_1 - \lambda_2) \frac{\delta}{\delta t} \exp[(1)\,ik]\,, \tag{2.8}$$

In view of equations of state (2.1) and (2.7), the equations of motion and continuity in a rotating coordinate system can be written in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\nabla P + \frac{1}{\varrho}\,\boldsymbol{j}\times\boldsymbol{B} + \nu\,\nabla^2\boldsymbol{v} - k_0^*\left[\frac{\partial}{\partial t}\,\nabla^2\boldsymbol{v} + 2(\boldsymbol{v}\cdot\nabla)\,\nabla^2\boldsymbol{v} - \nabla^2\{(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\}\right], \tag{2.9}$$

$$\operatorname{div}\boldsymbol{v} = 0\,, \tag{2.10}$$

where $\boldsymbol{\Omega}$ is the rotation vector, P is the pressure including the centrifugal term, $\nu = \eta_0/\varrho$ is the kinematic viscosity, $\boldsymbol{j}$ is the current density, $\boldsymbol{B}$ is the total magnetic field, $k_0^* = k_0/\varrho$ is the coefficient of elasticity and ϱ is the density of the liquid.

Neglecting displacement currents, the Maxwell equations and the generalized Ohm's law are

$$\operatorname{div}\boldsymbol{B} = 0\,, \qquad \operatorname{curl}\boldsymbol{B} = \mu\boldsymbol{J}\,, \qquad \operatorname{curl}\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}\,, \tag{2.11}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\,, \tag{2.12}$$

where $\boldsymbol{E}$ is the electric field, μ is the magnetic permeability and σ is the electrical conductivity of the liquid.

3. - The initial value formulation.

We consider the unsteady motion engendered in a semi-infinite, incompressible, electrically conducting Walters liquid B' bounded by an infinite rigid plate at $z = 0$. Both the liquid and the plate are in a state of rigid body rotation with a uniform angular velocity Ω about the z-axis normal to the plate. A uniform magnetic field $B_0 = (0, 0, B_0)$ parallel to the z-axis is applied to the rotating system. We examine the unsteady hydromagnetic flow due to an elliptic harmonic oscillations of the plate in its own plane.

We take the rectangular cartesian coordinate axes (x, y, z) such that the x, y axes lie in the plane of the plate. We seek a solution for the two dimensional flow field v so that it depends on z and t alone.

Making reference to work of Debnath $[1]_1$-$[1]_2$ based on the assumptions of small magnetic Reynolds number and negligible induced magnetic field, it turns out that the convective non-linear term in (2.9) disappears automatically. Consequently, the basic field equation for $\boldsymbol{v}(z, t)$ assumes the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + 2\Omega \boldsymbol{k} \times \boldsymbol{v} = \left(\nu - k_0^* \frac{\partial}{\partial t}\right) \frac{\partial^2 \boldsymbol{v}}{\partial z^2} + n^* \boldsymbol{v} , \tag{3.1}$$

where $n^* = (\sigma/\varrho)B_0^2$ represents the hydromagnetic parameter and has the same dimension as ν, and $\boldsymbol{k}$ is the unit vector parallel to the z-axis.

We assume that the superimposed oscillations of the plate are given by

$$u + iv = Uf(t) \qquad \text{on} \quad z = 0 , \; t > 0 , \tag{3.2}$$

where U is constant and has dimension of velocity, $f(t)$ is some physically realizable function of t, and u, v are the components of the velocity field.

As there is no disturbance at infinity, the boundary condition there is

$$u + iv \to 0 \qquad \text{as } z \to \infty , \; t > 0 . \tag{3.3}$$

The initial condition of the problem is

$$u + iv = 0 \qquad \text{at } t \leqslant 0 \text{ for all } z > 0 . \tag{3.4}$$

4. - The solution of the problem.

For convenience, we introduce non-dimensional variables z', t', u', v' defined by

$$z' = \frac{z}{D}, \quad t' = \Omega t, \quad (u', v') = \frac{1}{U}(u, v), \quad \text{where } D = \left(\frac{\nu}{\Omega}\right)^{1/2}.$$

In terms of these quantities, equation (3.1) can be written as, dropping the primes,

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2}\right) \boldsymbol{v} + k \frac{\partial^3 \boldsymbol{v}}{\partial z^2 \partial t} + 2\boldsymbol{k} \times \boldsymbol{v} + n\boldsymbol{v} = 0, \tag{4.1}$$

where

$$k = \frac{k_0^*}{D^2} \tag{4.2$_a$}$$

and

$$n = \frac{n^*}{\Omega} = \frac{\sigma}{\varrho} \frac{B_0^2}{\Omega}, \ . \tag{4.2$_b$}$$

In terms of a new notation $q \equiv u + iv$, equation (4.1), the boundary and the initial conditions can be written as

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2}\right) q + k \frac{\partial^3 q}{\partial t \, \partial z^2} + (n + 2i) q = 0, \tag{4.3}$$

$$q = f(t) \qquad \text{on} \quad z = 0, \ t > 0, \tag{4.4}$$

$$q \to 0 \qquad \text{as} \quad z \to \infty, \ t > 0, \tag{4.5}$$

$$q = 0 \qquad \text{at} \quad t \leqslant 0 \ \text{for all} \ z > 0. \tag{4.6}$$

To determine the principal features of the flow field and the structure of the boundary layers, it is of interest to consider the following cases:

(i) $f(t) = a \exp(i\omega' t) + b \exp(-i\omega' t)$, (ii) $f(t) = \exp(i\omega' t - m' t)$,

where a, b are complex constants, $m' = m/\Omega > 0$, and $\omega' = \omega/\Omega$ is the non-dimensional frequency of oscillations. As before, we shall drop the primes.

The initial value problem can readily be solved by means of the Laplace transform defined by the integral [**5**]

$$\bar{q}(z, s) = \int_0^\infty \exp[-st]\, q(z, t)\, \mathrm{d}t \, . \tag{4.7}$$

In view of this transform, the solution of (4.3) subject to the boundary and initial conditions can be expressed as the inverse Laplace integral

$$q(z, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(s) \left(1 - \frac{1}{2} skmz\right) \exp(st - m_1 z)\, \mathrm{d}s \, , \tag{4.8}$$

where $c > 0$ and $m_1 = (s + n + 2i)^{1/2}$.

Making reference to the table of the Laplace transform due to Campbell and Foster [**6**], the inversion complex integral (4.8) can be evaluated exactly and the solution for case (i) is given by

$$\begin{aligned} q(z, t) = {} & \frac{a}{2} \exp[i\omega t] \left[\left(1 - \frac{i}{2} \lambda_1 \omega kz\right) \exp(-\lambda_1 z) \operatorname{erfc}\left(\frac{z}{2\sqrt{t}} - \lambda_1 \sqrt{t}\right) + \right. \\ & \left. + \left(1 + \frac{i}{2} \lambda_1 \omega kz\right) \exp(\lambda_1 z) \operatorname{erfc}\left(\frac{z}{2\sqrt{t}} + \lambda_1 \sqrt{t}\right)\right] + \\ & + \frac{b}{2} \exp(-i\omega t) \left[\left(1 + \frac{i}{2} \omega \lambda_2 kz\right) \exp(-\lambda_2 z) \operatorname{erfc}\left(\frac{z}{2\sqrt{t}} - \lambda_2 \sqrt{t}\right) + \right. \\ & \left. + \left(1 - \frac{i}{2} \omega \lambda_2 kz\right) \exp(\lambda_2 z) \operatorname{erfc}\left(\frac{z}{2\sqrt{t}} + \lambda_2 \sqrt{t}\right)\right] + \\ & + \frac{k(a+b)z}{4\sqrt{\pi t^3}} \left(1 - \frac{z^2}{2t}\right) \exp\left(-2it - \frac{z^2}{4t}\right) + \\ & + \frac{ikz\omega}{2\sqrt{\pi t}} (b - a) \exp\left(-2it - \frac{z^2}{4t}\right) , \end{aligned} \tag{4.10}$$

where

$$\lambda_1 = \{n + i(2 + \omega)\}^{1/2} = (2 + \omega)^{1/2} (\alpha + i\beta) \, , \tag{4.11$_a$}$$

$$\lambda_2 = \{n + i(2 - \omega)\}^{1/2} = (2 - \omega)^{1/2} (\alpha - i\beta) \, . \tag{4.11$_b$}$$

With

$$\alpha = \{(\gamma^2+1)^{1/2}+\gamma\}^{1/2}\,, \qquad \beta = \{(\gamma^2+1)^{1/2}-\gamma\}^{1/2}\,, \qquad \gamma = \frac{n}{(2\pm\omega)}\,,$$

and erfc (x) is the usual complementary error function defined by the integral [5]

$$\text{erfc}\,(x) = 1-\text{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp\,[-\theta^2]\,d\theta\,. \tag{4.12}$$

Solution (4.10) describes a general feature of the unsteady boundary layer solution in a non-Newtonian rotating fluid in the presence of a uniform magnetic field.

It is worth noting that the effects of the Coriolis, hydromagnetic and elastic parameters are reflected on solution (4.10). In the absence of elastic parameter k, the velocity field (4.10) reduces identically to that obtained by Debnath $[1]_1$. In particular, when $\omega = 0$ and $a+b=1$ without any external magnetic field, result (4.10) becomes identical with that of Puri and Kulshrestha [7].

In the asymptotic limit $t\to\infty$, the ultimate steady state solution can readily be recovered from (4.10) by using the asymptotic representation of the complementary error function and has the form

$$q(z,t)\sim a\left(1-\frac{i}{2}\,\lambda_1\omega kz\right)\exp\,(i\omega t-\lambda_1 z)+ \\ +b\left(1+\frac{i}{2}\,\lambda_2\,\omega kz\right)\exp\,(-i\omega t-\lambda_2 z)\,. \tag{4.13}$$

The above result indicates the existence of composite Ekman-Stokes-Hartmann boundary layers of thicknesses of the order $\{2\nu/(2\Omega\pm\omega)\}^{1/2}(1/\alpha)$ in dimensional units. These layers lie between the Stokes-Ekman layers of thicknesses $(2\nu/(2\Omega\pm\omega))^{1/2}$ (as $\alpha\to 0$) and the Hartmann layer of thickness $(\nu/n)^{1/2}$ (as $\alpha\to\infty$). These results ensure that the inherent resonant type effect involved in the classical Stokes-Ekman problem is no longer present in this analysis.

It may be noted that since the present analysis is based upon the first order approximation to the elastic parameter k, the thicknesses of the boundary layers remain unaffected by k. However, with higher order approximation to k, the above solution can be refined so that the structures of the solution as well as the associated boundary layers are appreciably modified by the elastic parameter. Some attention is given to this point in a subsequent section.

Finally, the above analysis includes the results of the corresponding Newtonian rotating flow as special cases.

5. - Surface traction at the plate.

The surface traction at the plate is given by

$$(5.1)\qquad (\tau_{xz}+i\tau_{yz})_{z=0}=\left[\frac{\partial}{\partial z}\left(1-k\frac{\partial}{\partial t}\right)q\right]=$$

$$=\frac{a}{2}\exp(i\omega t)\left[2\lambda_1\sqrt{i}\,(C_1-iS_1)-\frac{2\exp(-\lambda_1^2 t)}{\sqrt{\pi t}}\right]+$$

$$+\frac{b}{2}\exp[-i\omega t]\left[2\lambda_2\sqrt{i}\,(C_2-iS_2)\,2-\frac{\exp(-\lambda_2^2 t)}{\sqrt{\pi t}}\right]+$$

$$+\frac{k}{2}a\exp[i\omega t]\left[\frac{2i\omega\exp(-\lambda_1^2 t)}{\sqrt{\pi t}}-\frac{\exp(-\lambda_1^2 t)}{\sqrt{\pi t^3}}-i\omega\lambda_1\sqrt{i}\,(C_1-iS_1)\right]+$$

$$+(-1)\frac{k}{2}\,b\exp[-i\omega t]\left[\frac{2i\omega\exp(-\lambda_2^2 t)}{\sqrt{\pi t}}+\frac{\exp(-\lambda_2^2 t)}{\sqrt{\pi t^3}}-i\omega\lambda_2\sqrt{i}\,(C_2-iS_2)\right]+$$

$$+\left[\frac{k(a+b)}{4\sqrt{\pi}t^{3/2}}+i\,\frac{k(b-a)\,\omega}{\sqrt{\pi t}}\right]\exp(-2it)\,,$$

where C_r, S_r $(r=1,2)$ are the well known Fresnel integrals of argument $(\lambda r/\sqrt{i})$ and $\lambda_{1,2}=\{(2\pm\omega)\,i+n\}^{1/2}$.

The surface traction is considerably modified by the elastic parameter k. For small value of t, (4.1) is unbounded for all k. This implies that the non-Newtonian fluid offers a greater resistence to the flow than the Newtonian fluid. On the other hand, the surface traction remains bounded for a large t. It can easily be verified that the corresponding result for Newtonian flow with or without magnetic field can readily be derived from (5.1) by putting $k=0$.

6. - The ultimate steady state solution.

In order to determine the effect of the elastic parameter on the structure of the boundary layers, it is important to refine the unsteady velocity field for sufficiently large times. The exact solution of the fundamental equation (4.3) subject to the boundary conditions (4.4)-(4.5) can be obtained by the Laplace transform technique in the form

$$q(z, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s) \exp\left(st - z\{(s+2i+n)(1+sk)\}^{1/2}\right) ds\,, \tag{6.1}$$

where

$$\bar{f}(s) = \frac{a}{s-i\omega} + \frac{b}{s+i\omega}\,. \tag{6.2}$$

The behaviour of the solution $q(z, t)$ for large time t can be determined from the transform solution $\bar{q}(z, s)$ for small s. Invoking this approximation in the integrand of (6.1), the integral (6.1) can be evaluated by using the table of Campbell and Foster [**6**]. The final closed form solution for $q(z, t)$ is then found as

$$\begin{aligned} q(z, t) = {} & \frac{a}{2}\exp[i\omega t]\Bigg[\exp\left(-z\{n - 2k\omega + i(2+\omega+nk\omega)\}^{1/2}\right) \\ & \operatorname{erfc}\left(\frac{z}{2}\left\{\frac{1+k(n+2i)}{t}\right\}^{1/2} - \left\{\frac{n-2k\omega+i(2+\omega+kn\omega)}{1+kn+2ik}\,t\right\}^{1/2}\right) \\ & + \exp\left(z\{n-2k\omega+i(2+\omega+kn\omega)\}^{1/2}\right) \\ & \operatorname{erfc}\left(\frac{z}{2}\left\{\frac{1+k(n+2i)}{t}\right\}^{1/2} + \left\{\frac{n-2k\omega+i(2+\omega+kn\omega)}{1+kn+2ik}\right\}^{1/2}\right)\Bigg] + \\ & + \frac{b}{2}\exp(-i\omega t)\Bigg[\exp\left(-z\{n+2k\omega+i(2-\omega-kn\omega)\}^{1/2}\right) \\ & \operatorname{erfc}\left(\frac{z}{2}\left\{\frac{1+k(n+2i)}{t}\right\}^{1/2} - \left\{\frac{n+2k\omega+i(2-\omega-kn\omega)}{1+kn+2ik}\,t\right\}^{1/2} \\ & + \exp\left(z\{n+2k\omega+i(2-\omega-kn\omega)\}^{1/2}\right) \\ & \operatorname{erfc}\left(\frac{z}{2}\left\{\frac{1+kn+2ik}{t}\right\}^{1/2} + \left\{\frac{n+2k\omega+i(2-\omega-kn\omega)}{1+kn+2ik}\,t\right\}^{1/2}\right]\,. \end{aligned} \tag{6.3}$$

Invoking the asymptotic nature of the complementary error function for large time, solution (6.3) is asymptotically equal to

$$(6.4)\qquad q(z,t)\sim a\exp(i\omega t)\exp\left(-z\{n-2k\omega+i(2+\omega+kn\omega)\}^{1/2}\right)+$$
$$+\,b\exp(-i\omega t)\exp\left(-z\{n+2k\omega+i(2-\omega-kn\omega)\}^{1/2}\right).$$

This can be put into a convenient and equivalent form

$$(6.5)\qquad q(z,t)\sim a\exp\left(i\omega t-(\alpha_1+i\beta_1)z\right)+b\exp\left(-i\omega t-(\alpha_2+i\beta_2)z\right)$$

where

$$(6.6)_a\qquad \alpha_1=\{(\gamma_1^2+1)^{1/2}+\gamma_1\}^{1/2}(2+\omega+kn\omega)^{1/2},$$

$$(6.6)_b\qquad \beta_1=\{(\gamma_1^2+1)^{1/2}-\gamma_1\}^{1/2}(2+\omega+kn\omega)^{1/2}\equiv\frac{1}{\alpha_1},$$

$$(6.7)_a\qquad \alpha_2=\{(\gamma_2^2+1)^{1/2}+\gamma_2\}^{1/2}(2-\omega-kn\omega)^{1/2},$$

$$(6.7)_b\qquad \beta_2=\{(\gamma_2^2+1)^{1/2}-\gamma_2\}^{1/2}(2-\omega-kn\omega)^{1/2}\equiv\frac{1}{\alpha_2},$$

$$(6.8)_a\qquad \gamma_1=\left(\frac{n-2k\omega}{2+\omega+kn\omega}\right),$$

$$(6.8)_b\qquad \gamma_2=\left(\frac{n+2k\omega}{2-\omega-kn\omega}\right).$$

Solution (6.5) reveals that it consists of two different Stokes-Ekman-Hartmann elastic boundary layers adjacent to the plate which has thicknesses of the order $1/\alpha_1$ and $1/\alpha_2$. In the absence of elastic parameter, these boundary layers become identical with those obtained by Debnath [**1**]. It is noted that these boundary layers are independently modified by k and n and remain bounded for all frequencies. This production is contrary to Thornley's [**1**]$_1$ findings for a Newtonian non-conducting rotating flow.

For a very weak magnetic field with a small value of the elastic parameter k, it is reasonable to neglect n^2, k^2 and nk so that

$$(\alpha_1+i\beta_1)\doteqdot\{n-2k\omega+i(2+\omega)\}^{1/2}\ \text{and}\ (\alpha_2+i\beta_2)\doteqdot\{n+2k\omega+i(2-\omega)\}^{1/2}.$$

With this approximation, the thicknesses of the Stokes-Ekman-Hartmann elastic boundary layers are of the order

$$\left\{\frac{\nu}{2\Omega+\omega-2k\omega+(\sigma/\varrho)\,B_0^2}\right\}^{1/2} \quad \text{and} \quad \left\{\frac{\nu}{2\Omega-\omega+2k\omega+(\sigma/\varrho)\,B_0^2}\right\}^{1/2}.$$

These results clearly suggest that the structure of the boundary layers is modified by all the parameters of the non-Newtonian hydromagnetic problem. This confirms that the Coriolis, hydromagnetic and elastic parameters have pronounced effects on the formation of the composite boundary layers. This is one of the stricking conclusions of the present analysis and has some physical interest. These findings are in accord with those obtained by Basu and Debnath [3] and have constrasting features with those of Puri and Kulshrestha [9].

7. - Concluding remarks.

It appears from the above mathematical analysis for the special case (i) that for case (ii) any new information or conclusion can hardly be given about the unsteady velocity field and the associated boundary layers. It may be reasonable to omit the calculation for case (ii) which includes case (i) when $m=0$ and the case considered by Puri and Kulshrestha [7] when $n=0$ and $\omega=0$. However, a mathematical analysis similar to that of case (i) may be reproduced for case (ii) without any difficulty.

It would be important and interesting to consider a more general non-torsionally or torsionally generated flow in a practical geometric configuration where an electrically conducting non-Newtonian fluid is bounded by two parallel disks or a closed container. These models appear to have physical significances for an understanding of geophysical problems and in particular the earth's liquid core motion. These problems may be reserved for future studies.

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References.

[1] L. DEBNATH: [•]$_1$ *On unsteady magnetohydrodynamic boundary layers in a rotating flow*, Z. Angew Math. Mech. **70** (1972), 1-8; [•]$_2$ *On the unsteady hydromagnetic boundary layer flow induced by torsional oscillations of a disk*, Appl. Sci. Res. (1972), (to appear).

[2] U. BASU and L. DEBNATH, *On unsteady non-Newtonian flowin a rotating system*, Pure and Applied Geophysics (1972), (to appear).

[3] K. WALTERS: [•]$_1$ IUTAM *International Symposium on second order effects in elasticity, elasticity and fluid dynamics*, Pergamon Press, New York 1964; [•]$_2$ *Second order effects in elasticity, plasticity and fluid dynamics*, Pergamon Press, New York 1964.

[4] J. G. OLDROYD, *On the formulation of rheological equations of state*, Proc. Roy. Soc. London **200** (1950), 523-541.

[5] I. N. SNEDDON, *The use of integral transforms*, Mc Graw Hill, Reading California 1972.

[6] G. A. CAMPBELL and R. M. FOSTER, *Fourier integrals for practical applications*, D. van Nostrand, New York 1948.

[7] P. PURI and P. K. KULSHRESTHA, *Rotation flow of non-newtonian fluids*, AMS Notice **17** (1970).

[8] C. THORNLEY, *On Stokes and Rayleigh layers in a rotating system*, Quart. J. Mech. Appl. Math. **21** (1948), 451-461.

Abstract.

Based upon the rheological equations of state for a class of elasticoviscous fluid, an initial value investigation is made of the hydromagnetic boundary layer flow generated in an incompressible, homogeneous, electrically conducting rotating fluid in the presence of a uniform magnetic field by the nontorsional elliptic harmonic oscillations of an infinite plate. Some qualitative and quantitative information is obtained about (i) *the unsteady velocity field and its important properties,* (ii) *the existence of the associated Stokes-Ekman-Hartmann - elastic boundary layers on the plate and* (iii) *the significant interaction of the Coriolis, hydromagnetic and elastic parameters. It is shown that there exists a combined Stokes-Ekman-Hartmann - elastic boundary layers of thicknesses of the order* $(\nu/2\Omega \pm \omega \mp 2k\omega + (\sigma/\varrho)B_0^2)^{\frac{1}{2}}$, *which remain bounded for both resonant and non-resonant cases. Attention is given to the behaviour of the transient solution for sufficiently large t. The surface traction at the plate is calculated and its general features are discussed. Several particular results are recovered as special cases of this analysis.*

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